Code Rate, Queueing Behavior and the Correlated Erasure Channel

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Motivation: Better Understanding Communication and Queueing

System Model
- Arrival Process at Transmitter
- Physical Layer and Coding Scheme
- Departure Process from Queue

Queueing Analysis
- Aggregate Model
- Equilibrium Distribution

Performance Evaluation
- Numerical Results

Discussion and Concluding Remarks
Digital Landscape and Delay-Sensitive Applications

Delay Tolerance and Rate Requirement

Context

- Point-to-point communication with limited feedback
- Packetized system supporting delay-sensitive traffic
- Reliability achieved through error-control coding
Target Infrastructures and Conceptual Model

Potential Scenarios

- Ad hoc and sensor networks
- Mesh networks
- Mobile Internet devices with real-time applications

Framework

- Complete queueing analysis
- Correlated erasure channel
- Specific structure with accurate characteristics

Three-Component System
Arrivals: Bernoulli Process

Discrete-Time Process
- Sampling period $N$
- Snapshots at onset of every codeword transmission

Packet Arrivals
- Queue length equals # of packets
- Arrival process is Bernoulli
- Probability of arrival $\gamma$
- Arrival process is IID
Arrivals: Packet Length Distribution

Packet Length Distribution
- Geometric # of bits
  \[ \Pr(L = \ell) = (1 - \rho)^{\ell-1}\rho \]
- IID over time
- Packets fragmented into segment of \( K \) bits

Data Fragments
- # of codewords per packet is \( M = \lceil L/K \rceil \)
- \( M \) is geometric
  \[ \Pr(M = m) = (1 - \rho_r)^{m-1}\rho_r \]
- Memoryless property
Physical Layer: Correlated Erasure Channel

Channel can be in two states
- **Good state** \( g \) in which bits arrive unaltered at destination
- **Bad state** \( b \) under which information is lost

**Markov Model**
- State at time \( t \) is \( C_t \)
- Channel evolution \( \{C_t\} \) is Markov
- Probability transition matrix

\[
P_t = \begin{bmatrix}
1 - \alpha & \alpha \\
\beta & 1 - \beta
\end{bmatrix}
\]
Coding Scheme: Random Binary Code

Code Length $N$

Information Bits $K$  Parity Bits $N - K$

Parity-Check Matrix
- Length of codeword, $N$
- # of information bits, $K$
- # of parity bits, $N - K$
- Generate random parity-check matrix $H$ of size $(N - K) \times N$

Code Properties
- Codebook is nullspace ($H$)
- ML decoding
- Code rate $r = K/N$ must be optimized
Coding Scheme: Probability of Decoding Failure

- Assume \( e \) erasures
- Parity check equation \( Hv = 0 \)
- Reorder elements

\[
\begin{bmatrix}
H_g & H_b
\end{bmatrix}
\begin{bmatrix}
v_g \\
v_b
\end{bmatrix} = H_g v_g + H_b v_b = 0
\]

- Uniquely codeword iff \( H_b \) has rank \( e \)

\[
H_b v_b = -H_g v_g
\]

- Probability decoding failure

\[
P_f(N - K, e) = 1 - \prod_{i=0}^{e-1} \left(1 - 2^i(N-K)\right)
\]
Decoding depends on \# of erasures \( E \) within codeword

\( E \) given by \# of visit to state \( b \)

Interested in conditional probabilities of type

\[
\Pr(E = e, C_{N+1} = d | C_1 = c)
\]

where \( e \in \mathbb{N}_0 \) and \( c, d \in \{b, g\} \)

Define matrix

\[
P_x = \begin{bmatrix}
(1 - \alpha)x & \alpha x \\
\beta & (1 - \beta)
\end{bmatrix}
\]

Let \([x^e]\) be operator on polynomials in \( x \)

\[
\Pr(E = e, C_{N+1} = d | C_1 = c) = [x^e]\begin{bmatrix} P_x^N \end{bmatrix}_{c,d}
\]
Aggregate Model and Transition Probabilities

- Queue length at onset of block $s$ is $Q_s$
- Channel state at same instant $C_{sN+1}$
- Queue length alone is **hidden Markov** but aggregate system is Markov
- State of augmented Markov state is
  \[ U_s = (C_{sN+1}, Q_s) \]

- Transition probability from $U_s$ to $U_{s+1}$
  \[
  \Pr(U_{s+1} = (d, q_{s+1})|U_s = (c, q_s)) = \sum_{e \in \mathbb{N}_0} \Pr(Q_{s+1} = q_{s+1}|E = e, Q_s = q_s) \times \Pr(E = e, C_{(s+1)N+1} = d|C_{sN+1} = c)
  \]
For $q \in \mathbb{N}$ and $c, d \in \{b, g\}$, let

\[
\mu_{cd} = \Pr(U_{s+1} = (d, q - 1) | U_s = (c, q))
\]
\[
\kappa_{cd} = \Pr(U_{s+1} = (d, q) | U_s = (c, q))
\]
\[
\lambda_{cd} = \Pr(U_{s+1} = (d, q + 1) | U_s = (c, q)).
\]

When queue is empty, use $\kappa_{cd}^0$ and $\lambda_{cd}^0$. 
Chapman-Kolmogorov Equations

For stable system, equilibrium distribution

$$\Pr(U = (c, q)) = \lim_{s \to \infty} \Pr(U_s = (c, q))$$

is unique solution to balance equations (plus boundary conditions)

$$(1 - \kappa_{gg})g_k = \lambda_{gg} g_{k-1} + \lambda_{bg} b_{k-1} + \kappa_{bg} b_k + \mu_{bg} b_{k+1} + \mu_{gg} g_{k+1}$$
$$(1 - \kappa_{bb})b_k = \lambda_{bb} b_{k-1} + \lambda_{gb} g_{k-1} + \kappa_{gb} g_k + \mu_{gb} g_{k+1} + \mu_{bb} b_{k+1}$$
Ordinary Generating Functions

Find solution to recurrence relation using transform methods

\[ B(z) = \sum_{k=0}^{\infty} b_k z^k \quad \quad G(z) = \sum_{k=0}^{\infty} g_k z^k. \]

Problem becomes equivalent to matrix equation

\[ A_1(z) \begin{bmatrix} B(z) \\ G(z) \end{bmatrix} = A_2(z) \begin{bmatrix} b_0 \\ g_0 \end{bmatrix} \]

where entries in \( A_1(z) \) and \( A_2(z) \) are quadratic polynomials.

- Closed-form solution exists
- Can be obtained using symbolic equation solvers
Performance Criteria

Admissible queue characterizations
- Average packet error rate
- Mean throughput
- Outage capacity

Criteria most relevant to delay-sensitive communications
- Probability of queue exceeding threshold
  \[ \Pr(Q \geq \tau) = \sum_{j=\tau}^{\infty} (b_j + g_j) \]
- Asymptotic decay in tail occupancy of queue
  \[ - \lim_{\tau \to \infty} \frac{1}{\tau} \log \Pr(Q \geq \tau) \]
System Parameters

Envisioned Scenario
Parameters selected to mimic operation of pico-cell aggregator for sensor network with GSM link

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = 114$</td>
<td>Code length</td>
</tr>
<tr>
<td>4.615 ms</td>
<td>Duration of transmission cycle</td>
</tr>
<tr>
<td>$\gamma = 0.25$</td>
<td>Probability of new packet arrival</td>
</tr>
<tr>
<td>$\rho = 1/175$</td>
<td>Mean packet size is 175 bits</td>
</tr>
<tr>
<td>9.48 Kbps</td>
<td>Average arrival rate</td>
</tr>
</tbody>
</table>

Erasure Channel

- Parameters are $\alpha = 0.08$ and $\beta = 0.02$
- Expected bit-erasure probability is 0.2
- Channel memory decays at rate $(1 - \alpha - \beta) = 0.9$
Optimal Code Rate Selection

Property I

Optimal code rate appears robust to threshold value

- \( \Pr(Q \geq \tau) \) for thresholds \( \tau \in \{5, 10, 15, 20, 25\} \)
- Minimum represents optimal operating point
- Uniformly achieved at \( K = 71 \)
- Best code rate is 0.62, capacity is 0.8
Tail Decay Rate and Arrival Process

- Tail decay rate
  \[ \lim_{\tau \to \infty} \frac{1}{\tau} \log \Pr(Q \geq \tau) \]

- Average arrival rate
  \[ \gamma/\rho_r \text{ per cycle} \]

- Maximum denotes optimal operating point

Property II

Optimal code rate seems robust to parameters of arrival process
Channel Parameters

<table>
<thead>
<tr>
<th>$1 - \alpha - \beta$</th>
<th>$K^*$</th>
<th>$\text{min } \Pr(Q &gt; 5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>80</td>
<td>0.049</td>
</tr>
<tr>
<td>0.5</td>
<td>76</td>
<td>0.085</td>
</tr>
<tr>
<td>0.9</td>
<td>71</td>
<td>0.368</td>
</tr>
<tr>
<td>0.98</td>
<td>87</td>
<td>0.598</td>
</tr>
<tr>
<td>0.99</td>
<td>114</td>
<td>0.246</td>
</tr>
</tbody>
</table>

- Channel memory affects optimal code rate
- It impacts queue distribution
- Limiting regime becomes packet erasure channel with memory

Property III

Optimal code rate is very sensitive to channel memory
Discussion and Concluding Remarks

Main Contributions

- New techniques to study communication for delay-sensitive traffic
- Correlated erasure channel provides insight into optimal code rate selection for short block lengths
- Optimal code rate appears linked to ratio between codeword time and coherence time

Avenues of Future Research

- Extend results to finite-state erasure channels
- Use framework to validate/invalidate common assumptions (fluid models, instantaneous capacity, etc.)
- Binary symmetric channels with memory
- Flexible block lengths
Thank You


