Code Rate, Queueing Behavior and the Correlated Erasure Channel

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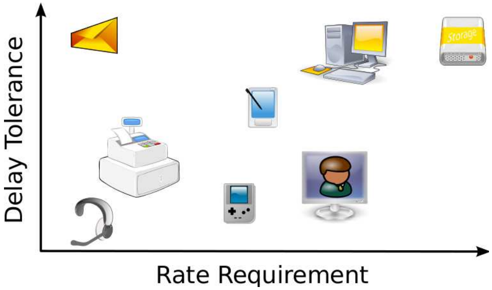
Digital Landscape and Delay-Sensitive Applications

Delay Tolerance and Rate Requirement



Context

- Point-to-point communication with limited feedback
- Packetized system supporting delay-sensitive traffic
- Reliability achieved through error-control coding

Target Infrastructures and Conceptual Model

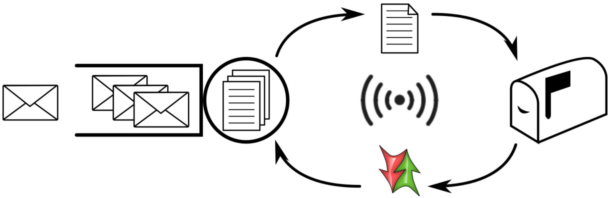
Potential Scenarios

- Ad hoc and sensor networks
- Mesh networks
- Mobile Internet devices with real-time applications

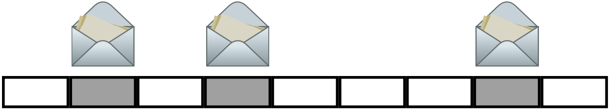
Framework

- Complete queueing analysis
- Correlated erasure channel
- Specific structure with accurate characteristics

Three-Component System



Arrivals: Bernoulli Process



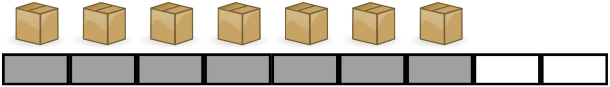
Discrete-Time Process

- Sampling period N
- Snapshots at onset of every codeword transmission

Packet Arrivals

- Queue length equals # of packets
- Arrival process is Bernoulli
- Probability of arrival γ
- Arrival process is IID

Arrivals: Packet Length Distribution



Packet Length Distribution

- Geometric # of bits

$$\Pr(L = \ell) = (1 - \rho)^{\ell - 1} \rho$$

- IID over time
- Packets fragmented into segment of K bits

Data Fragments

- # of codewords per packet is $M = \lceil L/K \rceil$
- M is geometric

$$\Pr(M = m) = (1 - \rho_r)^{m-1} \rho_r$$

- Memoryless property

Physical Layer: Correlated Erasure Channel

Channel can be in two states

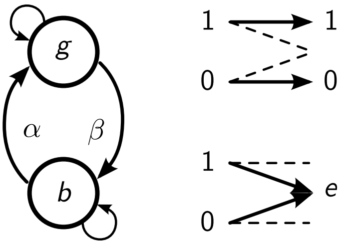
- Good state g in which bits arrive unaltered at destination
- Bad state b under which information is lost

Markov Model

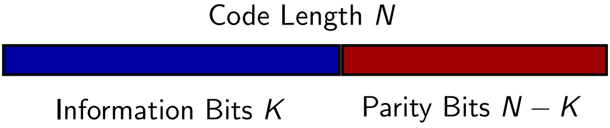
- State at time t is C_t
- Channel evolution $\{C_t\}$ is Markov
- Probability transition matrix

$$P_{\mathrm{t}} = \left[\begin{array}{cc} 1-\alpha & \alpha \\ \beta & 1-\beta \end{array} \right]$$

Channel Diagram



Coding Scheme: Random Binary Code



Parity-Check Matrix

- Length of codeword, N
- # of information bits, K
- # of parity bits, $N - K$
- Generate random parity-check matrix H of size $(N - K) \times N$

Code Properties

- Codebook is nullspace (H)
- ML decoding
- Code rate $r = K/N$ must be optimized

Coding Scheme: Probability of Decoding Failure

- Assume e erasures
- Parity check equation $H\mathbf{v} = \mathbf{0}$
- Reorder elements

$$\begin{bmatrix} H_g & H_b \end{bmatrix} \begin{bmatrix} \mathbf{v}_g \\ \mathbf{v}_b \end{bmatrix} = H_g \mathbf{v}_g + H_b \mathbf{v}_b = \mathbf{0}$$

- Uniquely codeword iff H_b has rank e

$$H_b \mathbf{v}_b = -H_g \mathbf{v}_g$$

- Probability decoding failure

$$P_{\mathrm{f}}(N-K, e) = 1 - \prod_{i=0}^{e-1} \left(1 - 2^{i-(N-K)}\right)$$

Departure: Distribution of Erasures

- Decoding depends on # of erasures E within codeword
- E given by # of visit to state b
- Interested in conditional probabilities of type

$$\Pr(E = e, C_{N+1} = d | C_1 = c)$$

  where $e \in \mathbb{N}_0$ and $c, d \in \{b, g\}$

- Define matrix

$$\mathbf{P}_x = \begin{bmatrix} (1-\alpha)x & \alpha x \\ \beta & (1-\beta) \end{bmatrix}$$

- Let $[\![x^e]\!]$ be operator on polynomials in x

$$\Pr(E = e, C_{N+1} = d | C_1 = c) = [\![x^e]\!] \left[\mathbf{P}_x^N\right]_{c,d}$$

Aggregate Model and Transition Probabilities

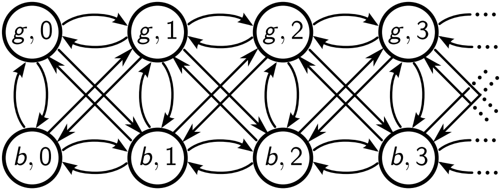
- Queue length at onset of block s is Q_s
- Channel state at same instant C_{sN+1}
- Queue length alone is hidden Markov but aggregate system is Markov
- State of augmented Markov state is

$$U_s = (C_{sN+1}, Q_s)$$

- Transition probability from U_s to U_{s+1}

$$\begin{aligned}
&\Pr(U_{s+1} = (d, q_{s+1}) | U_s = (c, q_s)) \\
&= \sum_{e \in \mathbb{N}_0} \Pr\left(Q_{s+1} = q_{s+1} | E = e, Q_s = q_s\right) \times \\
&\qquad \Pr\left(E = e, C_{(s+1)N+1} = d | C_{sN+1} = c\right)
\end{aligned}$$

State Diagram of Aggregate System



For $q \in \mathbb{N}$ and $c, d \in \{b, g\}$, let

$$\begin{aligned}
\mu_{cd} &= \Pr(U_{s+1} = (d, q-1) | U_s = (c, q)) \\
\kappa_{cd} &= \Pr(U_{s+1} = (d, q) | U_s = (c, q)) \\
\lambda_{cd} &= \Pr(U_{s+1} = (d, q+1) | U_s = (c, q)).
\end{aligned}$$

When queue is empty, use κ^0_{cd} and λ^0_{cd}.

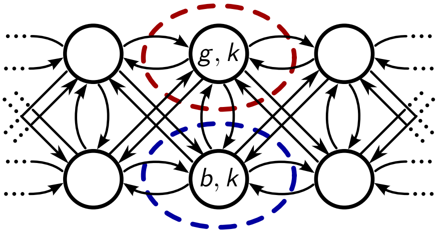
Chapman-Kolmogorov Equations

For stable system, equilibrium distribution

$$\Pr(U = (c, q)) = \lim_{s \to \infty} \Pr(U_s = (c, q))$$

is unique solution to balance equations (plus boundary conditions)

$$(1 - \kappa_{gg})g_k = \lambda_{gg} g_{k-1} + \lambda_{bg} b_{k-1} + \kappa_{bg} b_k + \mu_{bg} b_{k+1} + \mu_{gg} g_{k+1}$$
$$(1 - \kappa_{bb})b_k = \lambda_{bb} b_{k-1} + \lambda_{gb} g_{k-1} + \kappa_{gb} g_k + \mu_{gb} g_{k+1} + \mu_{bb} b_{k+1}$$



Ordinary Generating Functions

Find solution to recurrence relation using transform methods

$$B(z) = \sum_{k=0}^{\infty} b_k z^k \qquad G(z) = \sum_{k=0}^{\infty} g_k z^k.$$

Problem becomes equivalent to matrix equation

$$A_1(z) \begin{bmatrix} B(z) \\ G(z) \end{bmatrix} = A_2(z) \begin{bmatrix} b_0 \\ g_0 \end{bmatrix}$$

where entries in $A_1(z)$ and $A_2(z)$ are quadratic polynomials.

- Closed-form solution exists
- Can be obtained using symbolic equation solvers

Performance Criteria

Admissible queue characterizations

- Average packet error rate
- Mean throughput
- Outage capacity

Criteria most relevant to delay-sensitive communications

- Probability of queue exceeding threshold

$$\Pr(Q \geq \tau) = \sum_{j=\tau}^{\infty} (b_j + g_j)$$

- Asymptotic decay in tail occupancy of queue

$$-\lim_{\tau \to \infty} \frac{1}{\tau} \log \Pr(Q \geq \tau)$$

System Parameters

Envisioned Scenario

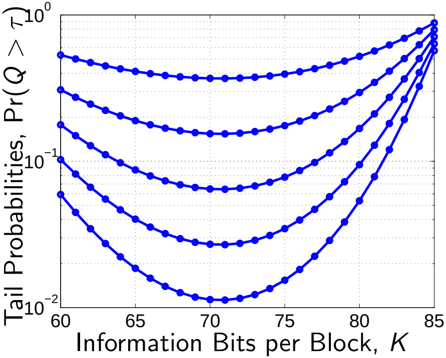
Parameters selected to mimic operation of pico-cell aggregator for sensor network with GSM link

$N = 114$	Code length
4.615 ms	Duration of transmission cycle
$\gamma = 0.25$	Probability of new packet arrival
$\rho = 1/175$	Mean packet size is 175 bits
9.48 Kbps	Average arrival rate

Erasure Channel

- Parameters are $\alpha = 0.08$ and $\beta = 0.02$
- Expected bit-erasure probability is 0.2
- Channel memory decays at rate $(1 - \alpha - \beta) = 0.9$

Optimal Code Rate Selection

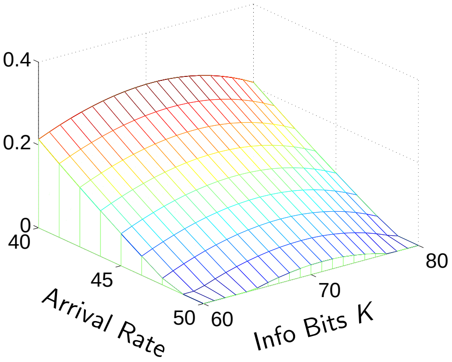


- $\Pr(Q > \tau)$ for thresholds $\tau \in \{5, 10, 15, 20, 25\}$
- Minimum represents optimal operating point
- Uniformly achieved at $K = 71$
- Best code rate is 0.62, capacity is 0.8

Property I

Optimal code rate appears robust to threshold value

Tail Decay Rate and Arrival Process



- Tail decay rate

$$-\lim_{\tau\to\infty}\frac{1}{\tau}\log\Pr(Q\geq\tau)$$

- Average arrival rate γ/ρ_r per cycle
- Maximum denotes optimal operating point

Property II

Optimal code rate seems robust to parameters of arrival process

Channel Parameters

$1-\alpha-\beta$	K^*	$\min \Pr(Q > 5)$
0	80	0.049
0.5	76	0.085
0.9	71	0.368
0.98	87	0.598
0.99	114	0.246

- Channel memory affects optimal code rate
- It impacts queue distribution
- Limiting regime becomes packet erasure channel with memory

Property III

Optimal code rate is very sensitive to channel memory

Discussion and Concluding Remarks

Main Contributions

- New techniques to study communication for delay-sensitive traffic
- Correlated erasure channel provides insight into optimal code rate selection for short block lengths
- Optimal code rate appears linked to ratio between codeword time and coherence time

Avenues of Future Research

- Extend results to finite-state erasure channels
- Use framework to validate/invalidate common assumptions (fluid models, instantaneous capacity, etc.)
- Binary symmetric channels with memory
- Flexible block lengths

Thank You

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