# The Serial Concatenation of Rate-1 Codes Through Uniform Random Interleavers

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# Outline

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- Serial Concatenation and Repeat-Accumulate (RA) Codes
- Serial Concatenation of Rate-1 Codes
- Repeat-Accumulate-Accumulate (RAA) Codes
- Summary

# Union Bounds on Performance

- Rate $r = k/n$, linear block code $C$
- Input Output Weight Enumerator Function (IOWEF) :

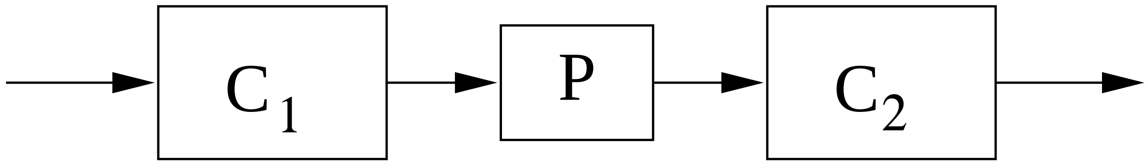
$$A_{w,h} \stackrel{\text{def}}{=} \#\text{ codewords, input weight } w\text{, output weight } h$$

- Union bound on word error probability $P_W$
(binary-input, memoryless channel, maximum-likelihood decoding):

$$P_W \leq \sum_{h=1}^{n} \sum_{w=1}^{k} A_{w,h} z^h$$

- $z$ is channel dependent; e.g., for Gaussian channel, $z = e^{-r(E_b/N_0)}$.
- For ensembles, replace $A_{w,h}$ by average IOWEF $\overline{A_{w,h}}$

# Serial Concatenation through a Uniform Interleaver



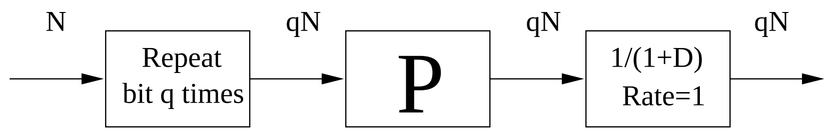

- Let $C_1$, $C_2$ be $(n_1, k_1)$, $(n_2, k_2)$ linear block codes with $n_1 = k_2$, and IOWEFs $A^{(1)}_{w,h}$, $A^{(2)}_{w,h}$.

- Let $C$ be the $(n_2, k_1)$ code obtained by serial concatenation of $C_1$ and $C_2$ through a uniform interleaver of size $n_1$, with average IOWEF $A_{w,h}$ :

$$A_{w,h} = \sum_{h_1=0}^{n_1} A^{(1)}_{w,h_1} \cdot \frac{A^{(2)}_{h_1,h}}{\binom{n_1}{h_1}}$$

# Repeat-Accumulate (RA) Codes

## (Divsalar, et al., Allerton'98)




- Repeat input block $x_1 x_2 \cdots x_N$ a total of $q$ times.
- Permute with random interleaver P of size $n = qN$.
- Accumulate over block:

$$u_1 u_2 \cdots u_n \;\longrightarrow\; v_1 v_2 \cdots v_n$$

$$v_1 = u_1$$

$$v_2 = u_1 + u_2$$

$$\vdots$$

$$v_n = u_1 + \cdots + u_n$$

# A Simple Example

- Rate-1, $(n, k) = (3, 3)$ Accumulate code:

| Input Bits | Input Weight $w$ | Output Bits | Output Weight $h$ |
|---|---|---|---|
| 000 | 0 | 000 | 0 |
| 001 | 1 | 001 | 1 |
| 010 | 1 | 011 | 2 |
| 100 | 1 | 111 | 3 |
| 011 | 2 | 010 | 1 |
| 101 | 2 | 110 | 2 |
| 110 | 2 | 100 | 1 |
| 111 | 3 | 101 | 2 |

# RA Weight Enumerator
## (Divsalar, et al., Allerton'98)

- $(qN, N)$ Repeat Code

$$A^{(1)}_{w,h} = \begin{cases} 0 & if \quad h \neq qw \\ \binom{qN}{w} & if \quad h = qw \end{cases}$$

$(n, n)$ Accumulate Code (cf. Öberg and Siegel, Allerton'98)

$$A^{(2)}_{w,h} = \binom{n-h}{\lfloor w/2 \rfloor} \binom{h-1}{\lceil w/2 \rceil - 1}$$

- $(qN, N)$ RA Code

$$\overline{A}_{w,h} = \frac{\binom{N}{w}\binom{qN-h}{\lfloor qw/2 \rfloor}\binom{h-1}{\lceil qw/2 \rceil - 1}}{\binom{qN}{qw}} \; if \; w, h > 0$$

# RA Coding Theorem
## (Divsalar, et al., Allerton'98)

- **Theorem:** For $q \geq 3$ , there exists $\gamma_q > 0$ such that, for any $E_b/N_0 > \gamma_q$, as the block length $N$ becomes large,

$$P_W^{UB} = O(N^{\beta}),$$

where $\beta = -\left\lceil \frac{q-2}{2} \right\rceil$. Hence $P_W \rightarrow \infty$ as $N \rightarrow \infty$.

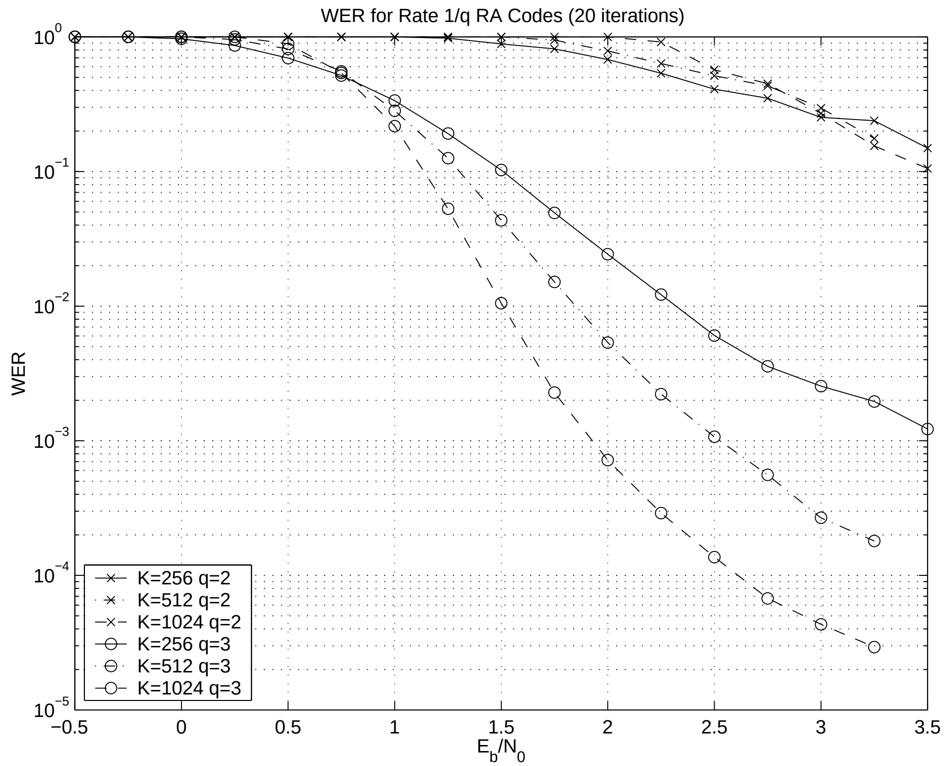
- Example $\gamma_q$ estimates (from Viterbi-Viterbi Bound):

$$\gamma_3 \approx 1.112 \text{ dB}$$

$$\gamma_4 \approx 0.313 \text{ dB}$$

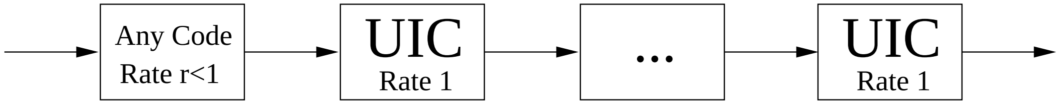
# Performance of RA Codes

- RA codes, $r = \frac{1}{3}$ and $r = \frac{1}{2}$, with iterative decoding.



# Serial Concatenation

- Consider an encoder architecture of the following form:




- UIC represents a Uniform Interleaver in cascade with a rate-1 Code (e.g., an $(n, n)$ Accumulate Code).

- We want to characterize the average IOWEF when the code is concatenated with $m$ UIC's.

# Serial Concatenation with UIC's

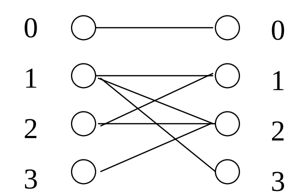
- Let IOWEF for rate-1, $(n, n)$ code be $A = [A_{i,j}], \;\; 0 \leq i, j \leq n$.

- Define IOW Transition Probabilities (IOWTP): $P = [P_{i,j}], \;\; 0 \leq i, j \leq n$,

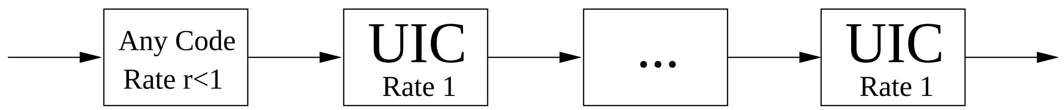
$$P_{i,j} = Pr(h = j | w = i) = \frac{A_{i,j}}{\binom{n}{i}}.$$

- Rate-1, $(n, k) = (3, 3)$ Accumulate code:

$$[P_{i,j}] = [Pr(h = j | w = i)] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.\overline{3} & 0.\overline{3} & 0.\overline{3} \\ 0 & 0.\overline{6} & 0.\overline{3} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



# Serial Concatenation with UIC's




- Outer $(n, k)$ code $C$, IOWEF $C_{w,h}$.
- Cascade of $m$, rate-1 $(n, n)$ UIC's, each with IOWTP $P = [P_{i,j}]$.
- Average IOWEF $\overline{A_{w,h}}$ of serial concatenation:

$$\overline{A_{w,h}} = \sum_{h_1=0}^{n} C_{w,h_1} \left[P^m\right]_{h_1,h}$$

# Perron-Frobenius Theory

- **Theorem:** An irreducible stochastic matrix $P$ has a unique stationary distribution $\pi$ s.t. $\pi P = \pi$ and $\sum \pi_i = 1$.

- **Theorem:** An irreducible aperiodic (primitive) non-negative matrix $P$ with unique stationary distribution $\pi$ satisfies:

$$\lim_{m \to \infty} P^m = \begin{bmatrix} \pi \\ \vdots \\ \pi \end{bmatrix}$$

- A linear rate-1 code is *primitive* if $P^+ = [P_{i,j}],\ i,\ j \neq 0,$ is a primitive matrix.

# A Simple Example (cont.)

- Rate-1, $(n, k) = (3, 3)$ Accumulate Code:

$$\begin{pmatrix} 0 & \frac{3}{7} & \frac{3}{7} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.\overline{3} & 0.\overline{3} & 0.\overline{3} \\ 0 & 0.\overline{6} & 0.\overline{3} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1+2}{7} & \frac{1+1+1}{7} & \frac{1}{7} \end{pmatrix}$$

$$\lim_{m \to \infty} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.\overline{3} & 0.\overline{3} & 0.\overline{3} \\ 0 & 0.\overline{6} & 0.\overline{3} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3/7 & 3/7 & 1/7 \\ 0 & 3/7 & 3/7 & 1/7 \\ 0 & 3/7 & 3/7 & 1/7 \end{pmatrix}$$

# Stationary Distributions and Rate-1 Codes

- **Proposition:** Consider the SC of $m$ primitive rate-1 codes through uniform interleavers. For non-zero input weights and as $m \to \infty$, the output weight distribution is independent of the input weight distribution and is

$$\pi_h = \frac{\binom{n}{h}}{2^n - 1}, \quad for \;\; h \neq 0.$$

- The ensemble averaged OWEF for any rate $r < 1$ code SC with $m \to \infty$ primitive rate-1 codes is

$$\overline{A_h} = \frac{2^{rn} - 1}{2^n - 1}\binom{n}{h}$$

# Iterated Rate-1 Codes Make Good Codes

- **Proposition:** For a code randomly chosen from an ensemble with average OWE $\overline{A_h}$,

$$Pr(d_{min} < d) \leq \sum_{h=1}^{d-1} \overline{A_{h.}}$$

(cf. Gallager, 1963)

- For length $n$, rate $r < 1$,and $\epsilon > 0$, let $d^*(n, r, \epsilon)$ be the largest value of $d$ satisfying

$$\sum_{h=0}^{d-1} \binom{n}{h} \leq \frac{2^n}{2^{rn} - 1}\epsilon.$$

# Iterated Rate-1 Codes Make Good Codes

- **Proposition:** For the ensemble of length-$n$ codes obtained by serial concatenation of a rate-$r$ linear block code with $m \to \infty$ primitive rate-1 linear codes through uniform random interleavers,

$$Pr(d_{min} < d^*(n, r, \epsilon)) \leq \epsilon.$$

- **Corollary:** Let $\epsilon = (2^{rn} - 1)/2^{rn}$. Then,

$$Pr(d_{min} < d^*) \leq \epsilon < 1,$$

where $d^*$is the largest value of $d$ satisfying

$$\sum_{h=0}^{d-1} \binom{n}{h} \leq 2^{n(1-r)} = 2^{n-k}$$

(i.e., at least one code satisfies the Gilbert-Varshamov Bound).

# Bounds for $CA^m$ Codes

- Rate r<1, outer code $C$.
- Serial concatenation with $m$ uniformly interleaved Accumulate codes.
- Compute largest value $d^*$ satisfying

$$\sum_{h=1}^{d-1} \overline{A_h} < \frac{1}{2}.$$

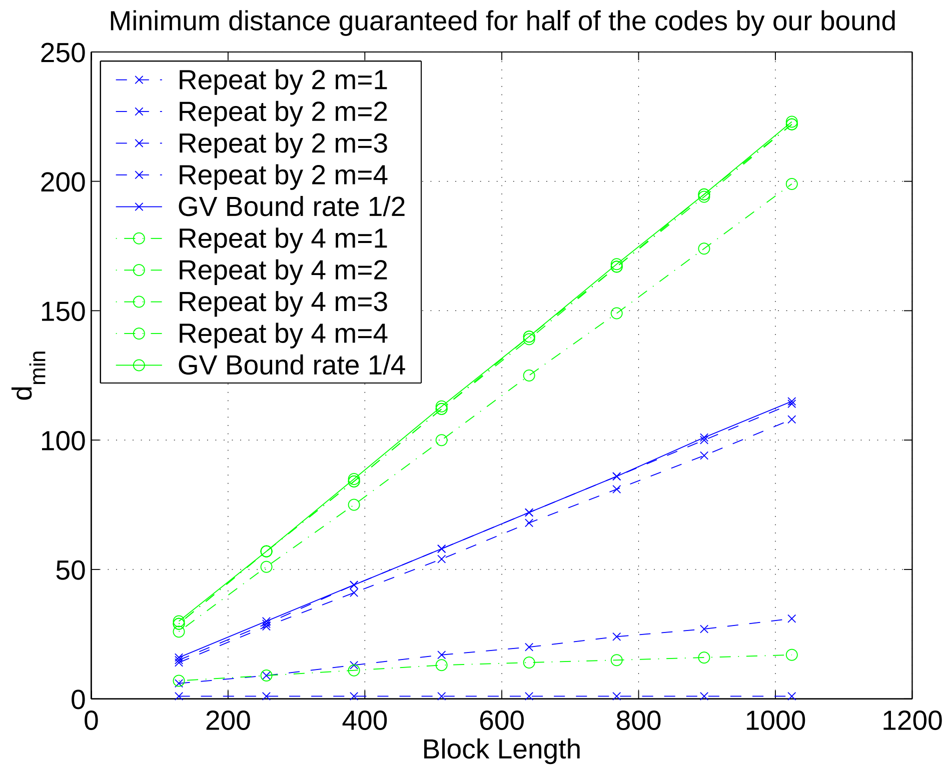
- Then, by the bound,

$$Pr(d_{min} < d^*) < \frac{1}{2},$$

(i.e., at least half the codes satisfy $d_{min} \geq d^*$).

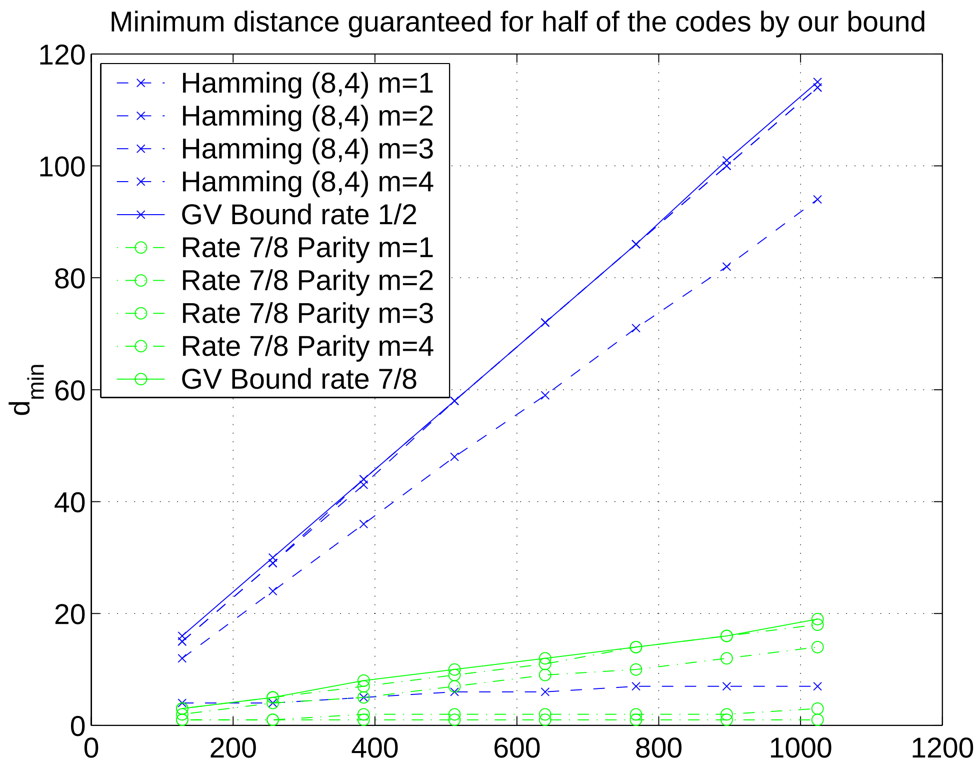
# Numerical Results

- $CA^m$ bounds for Repeat-2 and Repeat-4 codes, $1 \le m \le 4$.

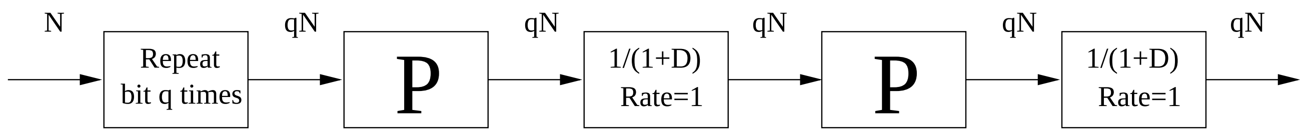


# Numerical Results

- For (8,4) Hamming and rate 7/8 parity-check codes, $1 \leq m \leq 4$.



# Rate 1/2, RAA Codes

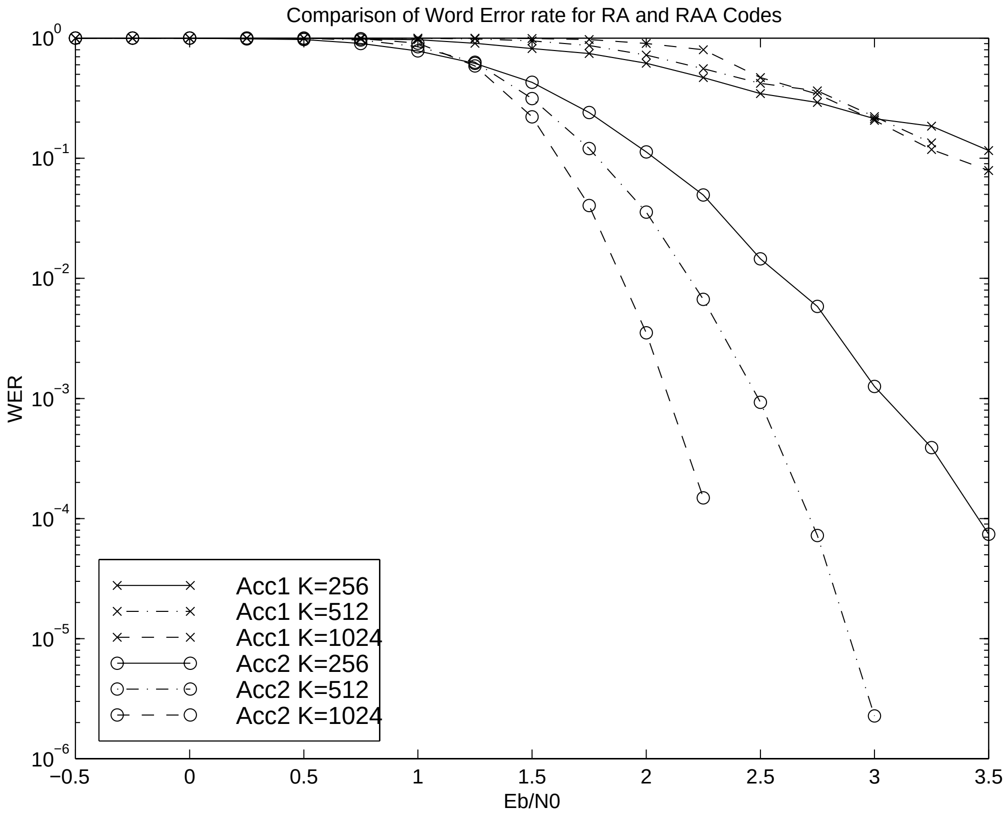



- For rate $r = 1/2$ RA code, analysis and simulations suggest that ***there is no word-error-probability interleaving gain***.

- For rate $r = 1/2$ RAA code, analysis and simulations suggest that ***there is...***

# Rate 1/2, RA vs. RAA

- Word-error-probability $P_W$ , from computer simulation.



# Summary

- We investigated a new class of codes based upon serial concatenation of a rate-$r$ code with $m \geq 1$, uniformly interleaved rate-1 codes.
- We analyzed the output weight enumerator function $\overline{A_h}$ for finite $m$, as well as asymptotically for $m \rightarrow \infty$.
- We evaluated the "goodness" of these ensembles in terms of the Gilbert-Varshamov Bound.
- We compared $r = 1/2$ RAA codes to RA codes.

**Question:** For $r = 1/2$, RAA codes, is there a $\gamma$ such that , for $E_b/N_0 > \gamma$ , $P_W \rightarrow 0$ as $N \rightarrow \infty$ ?