The Serial Concatenation of Rate-1 Codes Through Uniform Random Interleavers

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Outline

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Union Bounds on Performance

- Rate $r = k/n$, linear block code $C$
- Input Output Weight Enumerator Function (IOWEF): 
  \[ A_{w,h} \overset{\text{def}}{=} \# \text{ codewords, input weight } w, \text{ output weight } h \]
- Union bound on word error probability $P_W$
  (binary-input, memoryless channel, maximum-likelihood decoding):
  \[ P_W \leq \sum_{h=1}^{n} \sum_{w=1}^{k} A_{w,h} z^h \]
- $z$ is channel dependent; e.g., for Gaussian channel, $z = e^{-r(E_b/N_0)}$.
- For ensembles, replace $A_{w,h}$ by average IOWEF $\overline{A_{w,h}}$
Let $C_1$, $C_2$ be $(n_1, k_1)$, $(n_2, k_2)$ linear block codes with $n_1 = k_2$, and IOWEFs $A^{(1)}_{w,h}$, $A^{(2)}_{w,h}$.

Let $C$ be the $(n_2, k_1)$ code obtained by serial concatenation of $C_1$ and $C_2$ through a uniform interleaver of size $n_1$, with average IOWEF $A_{w,h}$:

$$A_{w,h} = \sum_{h_1=0}^{n_1} A^{(1)}_{w,h_1} \cdot \frac{A^{(2)}_{h_1,h}}{\binom{n_1}{h_1}}$$
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Repeat-Accumulate (RA) Codes
(Divsalar, et al., Allerton’98)

1/(1+D)
Rate=1

- Repeat input block $x_1 x_2 \cdots x_N$ a total of $q$ times.
- Permute with random interleaver $P$ of size $n = qN$.
- Accumulate over block:

$$u_1 u_2 \cdots u_n \rightarrow v_1 v_2 \cdots v_n$$

$$v_1 = u_1$$

$$v_2 = u_1 + u_2$$

$$\vdots$$

$$v_n = u_1 + \cdots + u_n$$
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A Simple Example

- Rate-1, \((n, k) = (3, 3)\) Accumulate code:

<table>
<thead>
<tr>
<th>Input Bits</th>
<th>Input Weight (w)</th>
<th>Output Bits</th>
<th>Output Weight (h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>000</td>
<td>0</td>
<td>000</td>
<td>0</td>
</tr>
<tr>
<td>001</td>
<td>1</td>
<td>001</td>
<td>1</td>
</tr>
<tr>
<td>010</td>
<td>1</td>
<td>011</td>
<td>2</td>
</tr>
<tr>
<td>100</td>
<td>1</td>
<td>111</td>
<td>3</td>
</tr>
<tr>
<td>011</td>
<td>2</td>
<td>010</td>
<td>1</td>
</tr>
<tr>
<td>101</td>
<td>2</td>
<td>110</td>
<td>2</td>
</tr>
<tr>
<td>110</td>
<td>2</td>
<td>100</td>
<td>1</td>
</tr>
<tr>
<td>111</td>
<td>3</td>
<td>101</td>
<td>2</td>
</tr>
</tbody>
</table>
RA Weight Enumerator
(Divsalar, et al., Allerton’98)

- \((qN, N)\) Repeat Code

\[
A^{(1)}_{w,h} = \begin{cases} 
0 & \text{if } h \neq qw \\
\left( \begin{array}{c} qN \\ w \end{array} \right) & \text{if } h = qw
\end{cases}
\]

- \((n, n)\) Accumulate Code (cf. Öberg and Siegel, Allerton’98)

\[
A^{(2)}_{w,h} = \left( n - h \right) \left( h - 1 \right)
\]

- \((qN, N)\) RA Code

\[
\overline{A}_{w,h} = \frac{\binom{N}{w} \binom{qN-h}{\lfloor qw/2 \rfloor} \binom{h-1}{\lfloor qw/2 \rfloor - 1}}{\binom{qN}{qw}} \text{ if } w, h > 0
\]
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RA Coding Theorem
(Divsalar, et al., Allerton’98)

- **Theorem**: For $q \geq 3$, there exists $\gamma_q > 0$ such that, for any $E_b/N_0 > \gamma_q$, as the block length $N$ becomes large,

$$P_W^{UB} = O(N^\beta),$$

where $\beta = -\lceil\frac{q-2}{2}\rceil$. Hence $P_W \to \infty$ as $N \to \infty$.

- **Example $\gamma_q$ estimates (from Viterbi-Viterbi Bound):**

  $\gamma_3 \approx 1.112$ dB

  $\gamma_4 \approx 0.313$ dB
RA codes, \( r = \frac{1}{3} \) and \( r = \frac{1}{2} \), with iterative decoding.
Serial Concatenation

- Consider an encoder architecture of the following form:

- UIC represents a Uniform Interleaver in cascade with a rate-1 Code (e.g., an \((n, n)\) Accumulate Code).

- We want to characterize the average IOWEF when the code is concatenated with \(m\) UIC’s.
Serial Concatenation with UIC's

- Let IOWEF for rate-1, \((n, n)\) code be \(A = [A_{i,j}], \ 0 \leq i, j \leq n.\)

- Define IOW Transition Probabilities (IOWTP): \(P = [P_{i,j}], \ 0 \leq i, j \leq n,

\[
P_{i,j} = Pr(h = j \mid w = i) = \frac{A_{i,j}}{\binom{n}{i}}.
\]

- Rate-1, \((n, k) = (3, 3)\) Accumulate code:

\[
[P_{i,j}] = [Pr(h = j \mid w = i)] = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0.3 & 0.3 & 0.3 \\
0 & 0.6 & 0.3 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]
Serial Concatenation with UIC’s

- Outer \((n, k)\) code \(C\), IOWEF \(C_{w,h}\).
- Cascade of \(m\), rate-1 \((n, n)\) UIC’s, each with IOWTP \(P = [P_{i,j}]\).
- Average IOWEF \(\overline{A}_{w,h}\) of serial concatenation:

\[
\overline{A}_{w,h} = \sum_{h_1=0}^{n} C_{w,h_1} \left[P^m\right]_{h_1,h}
\]
Perron-Frobenius Theory

- **Theorem:** An irreducible stochastic matrix $P$ has a unique stationary distribution $\pi$ s.t. $\pi P = \pi$ and $\sum \pi_i = 1$.

- **Theorem:** An irreducible aperiodic (primitive) non-negative matrix $P$ with unique stationary distribution $\pi$ satisfies:

$$
\lim_{m \to \infty} P^m = \begin{bmatrix} \pi \\ \vdots \\ \pi \end{bmatrix}
$$

- A linear rate-1 code is *primitive* if $P^+ = [P_{i,j}]$, $i \neq j \neq 0$, is a primitive matrix.
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A Simple Example (cont.)

- Rate-1, \((n, k) = (3, 3)\) Accumulate Code:

\[
\begin{pmatrix}
0 & 3/7 & 3/7 & 1/7
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0.3 & 0.3 & 0.3 \\
0 & 0.6 & 0.3 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 1/7 & 1+1+1/7 & 1/7
\end{pmatrix}
\]

\[
\lim_{m \to \infty} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0.3 & 0.3 & 0.3 \\
0 & 0.6 & 0.3 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}^m = \begin{pmatrix}
0 & 3/7 & 3/7 & 1/7 \\
0 & 3/7 & 3/7 & 1/7 \\
0 & 3/7 & 3/7 & 1/7
\end{pmatrix}
\]
• Proposition: Consider the SC of \( m \) primitive rate-1 codes through uniform interleavers. For non-zero input weights and as \( m \to \infty \), the output weight distribution is independent of the input weight distribution and is

\[
\pi_h = \frac{{n \choose h}}{2^{n-1}}, \quad \text{for } h \neq 0.
\]

• The ensemble averaged OWEF for any rate \( r < 1 \) code SC with \( m \to \infty \) primitive rate-1 codes is

\[
\overline{A}_h = \frac{2^{rn} - 1}{{2n - 1 \choose h}}
\]
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Iterated Rate-1 Codes Make Good Codes

• **Proposition:** For a code randomly chosen from an ensemble with average OWE $A_h$,

$$Pr(d_{\text{min}} < d) \leq \sum_{h=1}^{d-1} A_h.$$  

(cf. Gallager, 1963)

• For length $n$, rate $r < 1$, and $\epsilon > 0$, let $d^*(n, r, \epsilon)$ be the largest value of $d$ satisfying

$$\sum_{h=0}^{d-1} \binom{n}{h} \leq \frac{2^n}{2^{rn} - 1} \epsilon.$$
Iterated Rate-1 Codes Make Good Codes

- **Proposition:** For the ensemble of length-$n$ codes obtained by serial concatenation of a rate-$r$ linear block code with $m \to \infty$ primitive rate-1 linear codes through uniform random interleavers,

$$Pr(d_{min} < d^*(n, r, \epsilon)) \leq \epsilon.$$

- **Corollary:** Let $\epsilon = (2^{rn} - 1)/2^{rn}$. Then,

$$Pr(d_{min} < d^*) \leq \epsilon < 1,$$

where $d^*$ is the largest value of $d$ satisfying

$$\sum_{h=0}^{d-1} \binom{n}{h} \leq 2^{n(1-r)} = 2^{n-k},$$

(i.e., at least one code satisfies the Gilbert-Varshamov Bound).
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Bounds for $C A^m$ Codes

- Rate $r < 1$, outer code $C$.
- Serial concatenation with $m$ uniformly interleaved Accumulate codes.
- Compute largest value $d^*$ satisfying

$$
\sum_{h=1}^{d-1} A_h < \frac{1}{2}.
$$

- Then, by the bound,

$$
Pr(d_{min} < d^*) < \frac{1}{2},
$$

(i.e., at least half the codes satisfy $d_{min} \geq d^*$).
Numerical Results

- $CA^m$ bounds for Repeat-2 and Repeat-4 codes, $1 \leq m \leq 4$. 

![Graph showing minimum distance guaranteed for half of the codes by our bound](image)
For (8,4) Hamming and rate 7/8 parity-check codes, $1 \leq m \leq 4$. 

![Graph showing minimum distance guaranteed for half of the codes by our bound]
For rate $r = 1/2$ RA code, analysis and simulations suggest that there is no word-error-probability interleaving gain.

For rate $r = 1/2$ RAA code, analysis and simulations suggest that there is...
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Rate 1/2, RA vs. RAA

- Word-error-probability $P_W$, from computer simulation.

Comparison of Word Error rate for RA and RAA Codes

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Summary

• We investigated a new class of codes based upon serial concatenation of a rate-\(r\) code with \(m \geq 1\), uniformly interleaved rate-1 codes.

• We analyzed the output weight enumerator function \(A_h\) for finite \(m\), as well as asymptotically for \(m \to \infty\).

• We evaluated the “goodness” of these ensembles in terms of the Gilbert-Varshamov Bound.

• We compared \(r = 1/2\) RAA codes to RA codes.

**Question:** For \(r = 1/2\), RAA codes, is there a \(\gamma\) such that, for \(E_b/N_0 > \gamma\),

\[ P_W \to 0 \text{ as } N \to \infty \]