Iterative Decoding of High Rate LDPC Codes With Applications in Compressed Sensing

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Outline

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1. Introduction
   - Compressed Sensing and Connections with Coding
   - Low-Density Parity-Check (LDPC) Codes
2. High Rate Scaling of Density Evolution
   - DE Scaling Analysis for the BEC
   - DE Scaling Analysis for the q-SC
3. High Rate Scaling Stopping Set Analysis
   - Scaling Law Analysis for Stopping Sets on the BEC
   - Scaling Law Analysis for Stopping Sets on the q-SC With LM1
4. Summary and Open Problems

Motivation for This Work

- Connections between coding and compressed sensing
- Analyze CS systems with tools from coding theory
- Analyze CS systems based on high rate scaling of LDPC codes

Compressed Sensing

- Compressed sensing (CS) is a relatively new area of signal processing and statistics that focuses on signal reconstruction from a small number of linear (i.e. dot product) measurements
- Use m dot-product samples to reconstruct a signal
  - The signal vector is $x \in \mathbb{R}^n$
  - The $m \times n$ measurement matrix is $\Phi \in \mathbb{R}^{m \times n}$
  - The length-m sample vector is $y = \Phi x$
- Given y, the valid signal set is $V(y) = \{x' \in \mathbb{R}^n | \Phi x' = y\}$
  - If $m < n$, then a unique solution is not possible
  - With prior knowledge, we try to choose a “good” solutions
  - If x is sparse (w.r.t. Hamming weight $\|\cdot\|_H$), then

$$\hat{x} = \arg \min_{x' \in V(y)} \|x'\|_H$$

Compressed Sensing and Coding

Compressed Sensing

- Signal: $x \in \mathbb{R}^n$
  - sparse: $\|x\|_H \leq \delta n$
- Measurement matrix:
  $\Phi \in \mathbb{R}^{m \times n}$
  - Blind to nullspace of Φ
- Sample vector: $y = \Phi x$
- Dec: $\hat{x} = \arg \min_{x' : y = \Phi x'} \|x'\|_H$

Coding

- Error pattern: $e \in F^n$
  - sparse: $\Pr(e_i \neq 0) = \delta$
- Parity-check matrix:
  $H \in F^{m \times n}$
  - Code is nullspace of H
- Syndrome: $s = He$
- Dec: $\hat{e} = \arg \min_{e' : s = He'} \|e'\|_H$

Analysis Tools and Reconstruction Measures

Compressed Sensing

- Uniform reconstruction (some Φ works form all x)
- Uniform-in-probability (For any x, random Φ works, prob. in Φ)
- Randomized reconstruction (work w.h.p. for random x and Φ, prob. in both x and Φ)

Tools from Coding

- Stopping set analysis ignoring false verification
- Stopping set analysis including false verification
- Density evolution

Outline

Low-Density Parity-Check (LDPC) Codes

- Linear codes with sparse parity-check matrix H
  - Codes over GF(q) defined by $H_{ij} \in$GF(q) s.t. $\sum_j H_{ij} x_j = 0$
- Bipartite graph representation
  - An edge connects check node i to symbol node j if $H_{ij} \neq 0$
  - Irregular codes defined by degree distributions $\lambda(x), \rho(x)$
- Message Passing (MP) Decoding
  - Nodes iteratively pass symbol estimates to one another
  - Density evolution (DE) analysis and stopping set analysis
    - For (3,6) codes, the DE threshold is 43%, SS threshold is 1.8%

Verification Based Decoding Algorithm

- Idea: Verify messages which are correct w.h.p.
- Algorithms by Luby and Mitzenmacher (Allerton02, IT05)
  - 1st Alg. (LM1): Verify all symbols if check sums to zero
  - 2nd Alg. (LM2): LM1 + Verify if two msgs match at symbol
- False verification: event that message verified but not correct
  - Assumption that the weighted sum of non-zero coefficients does not equal to zero
  - can be avoided by assumption or by randomizing Φ
- These algorithms can be used in CS system over real numbers since the assumption “verified symbols are correct w.h.p.” holds equally well over the $GF(q)$ with large q and the real numbers.

CS Reconstruction via LM1

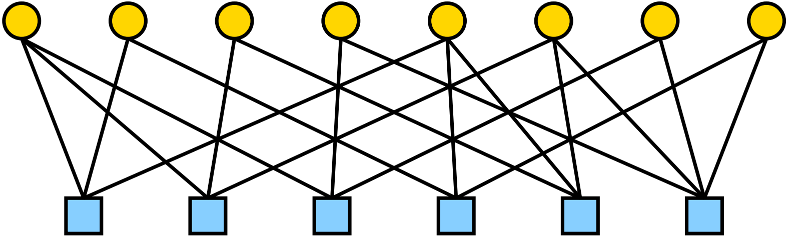
Example

Let Φ be the following parity-check matrix H

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

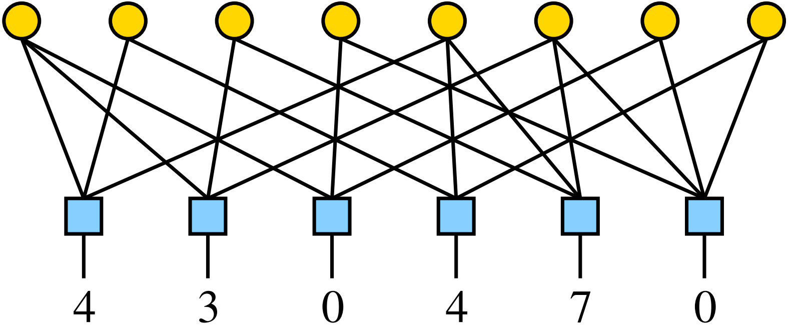
Note: We choose edge weights of one for simplicity

CS Reconstruction via LM1



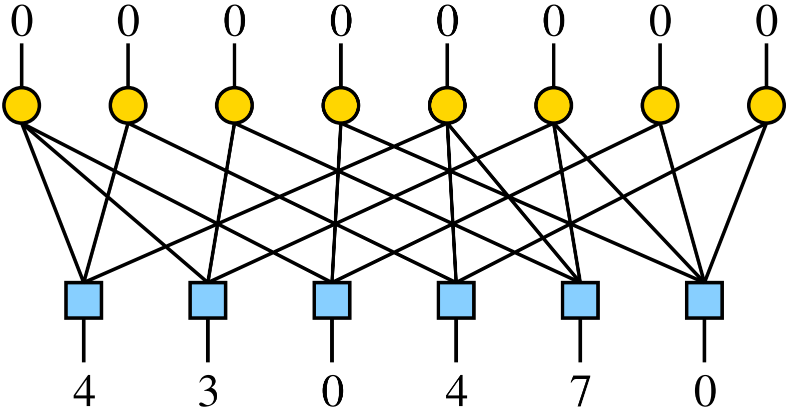
Signal (circles) measurement (squares) model

CS Reconstruction via LM1



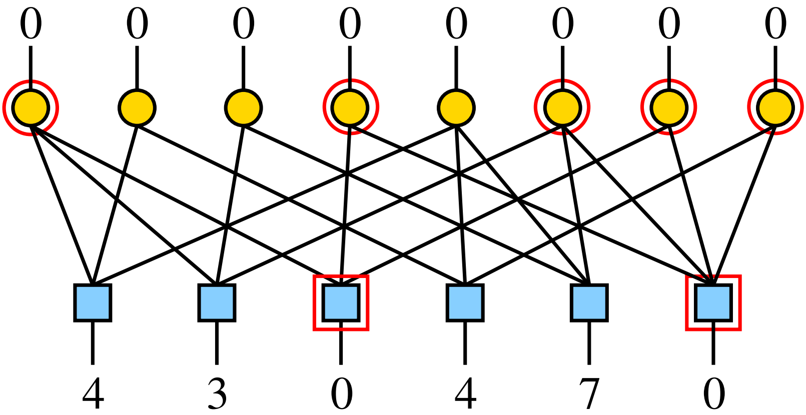
With measurements exposed

CS Reconstruction via LM1



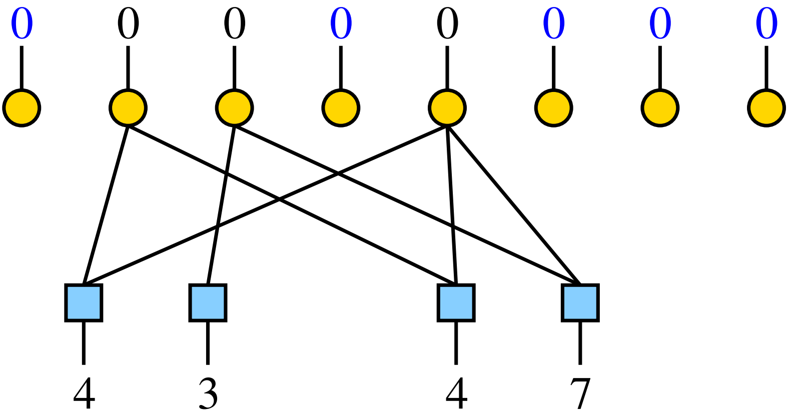
All-zero signal assumed for decoding

CS Reconstruction via LM1



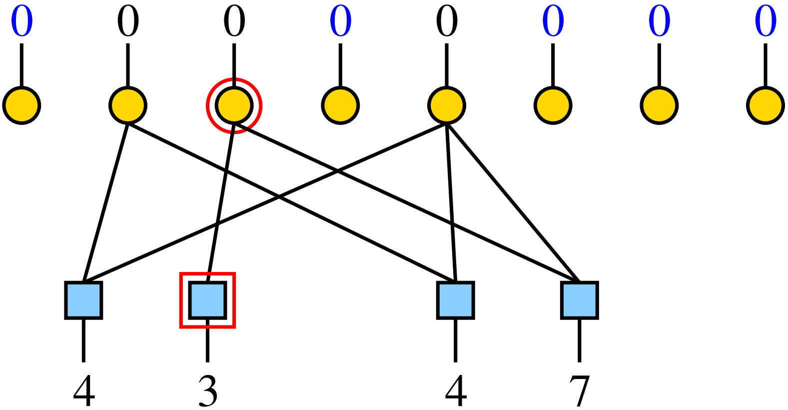
Assume all satisfied checks are correct

CS Reconstruction via LM1



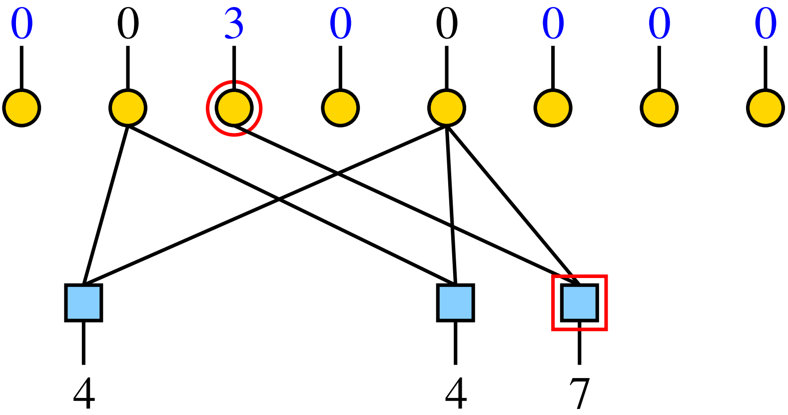
Remove edges and fix values

CS Reconstruction via LM1



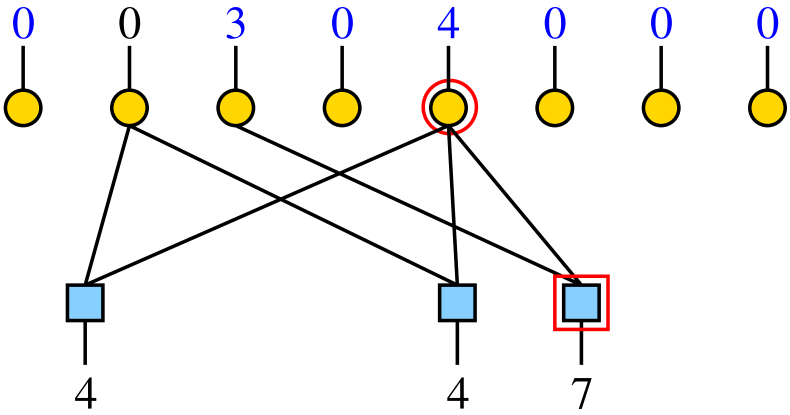
Use degree-1 check to determine a variable

CS Reconstruction via LM1



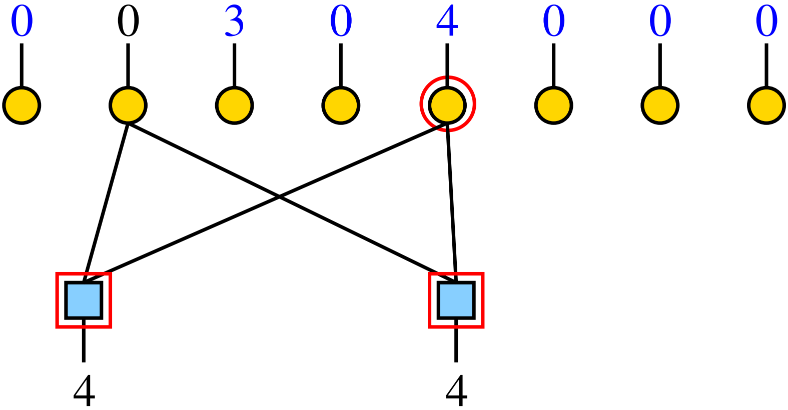
This value can be removed from all equations

CS Reconstruction via LM1



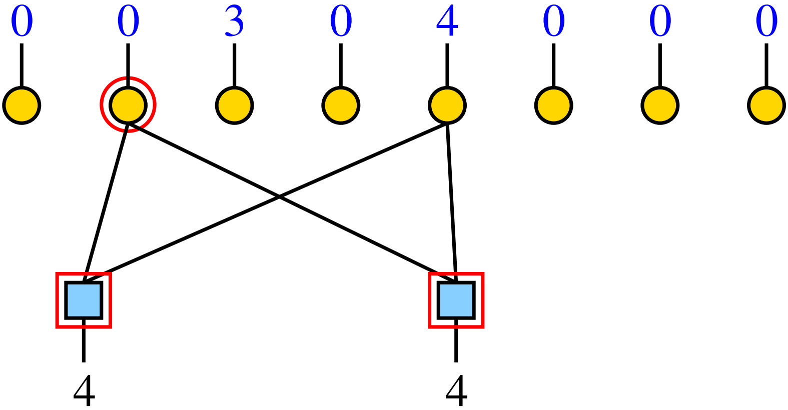
Another variable is determined

CS Reconstruction via LM1



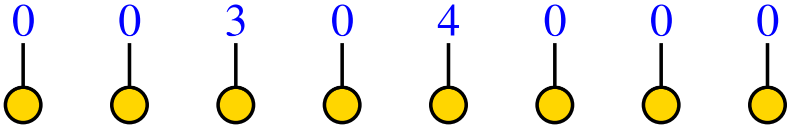
This value can be removed from all equations

CS Reconstruction via LM1



Final value is determined in two ways

CS Reconstruction via LM1



Reconstruction is successful

Outline

Introduction to DE and Scaling Law Analysis

- Given a code ensemble, DE tracks the evolution of the distribution of the message on the graph as decoding proceeds.
- DE gives the threshold below which the decoding succeeds w.h.p. as the number of iterations goes to infinity.
- Consider a sequence of ensembles, scaling law analysis shows how the thresholds scale with the parameters of the ensembles.

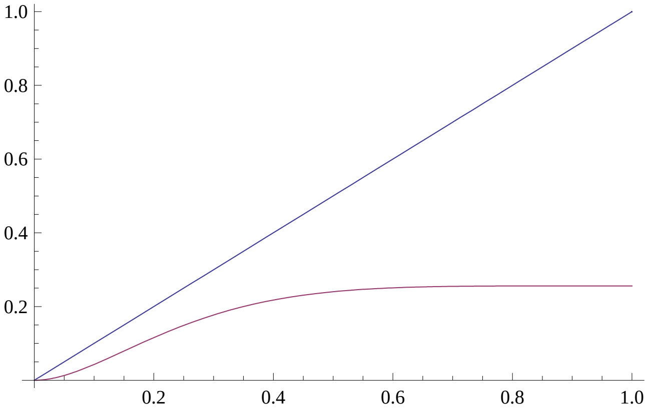
Example 1:

A (j, k) code with cE check nodes can recover E erasures w.h.p. (the number of check nodes scales linearly with the number of erasures).

Example 2:

A (j, k) code with LM2 decoding can recover a K-sparse signal with only cK measurements for randomized reconstruction.

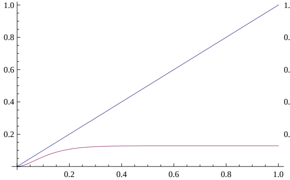
An Example for Scaling Analysis



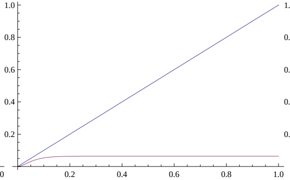
Unscaled DE equation: $y = \delta_0(1 - (1-x)^5)^2$

What's the DE curve like when rate$\rightarrow$ 1, or, fix j and let $k \rightarrow \infty$?

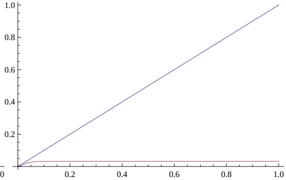
An Example for Scaling Law Analysis



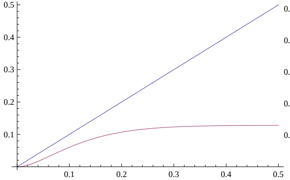
$\delta = \delta_0/2$, $k = 2k_0$



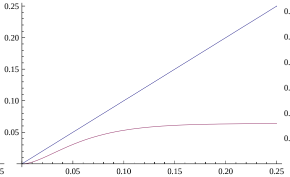
$\delta = \delta_0/4$, $k = 4k_0$



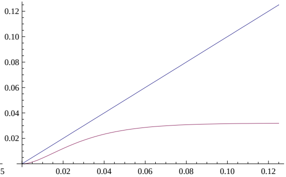
$\delta = \delta_0/8$, $k = 8k_0$



scaled by $x \leftarrow x/2$



scaled by $x \leftarrow x/4$



scaled by $x \leftarrow x/8$

DE Scaling Analysis for the BEC

- Consider the sequence of (j, k) regular ensemble.
- Fixed j, as $k \to \infty \Rightarrow$ rate $\to 1$
- $\overline{\alpha}_j \triangleq \sup \left\{ \alpha : \lambda \left(1 - e^{-\alpha j x}\right) \leq x, \text{ for } x \in (0, 1] \right\}$
- Let $\delta = \frac{\alpha j}{(k-1)}$, the iterative decoding fails (w.h.p as $n \to \infty$) for all k if $\alpha > \overline{\alpha}_j$.
- Conversely, if $\alpha < \overline{\alpha}_j$, then $\exists K < \infty$ such that iterative decoding succeeds (w.h.p as $n \to \infty$) for all $k \geq K$.

Remark

$\bar{\alpha}_j$ means the fraction of the capacity that can be achieved. For (j, k) regular LDPC codes, $\overline{\alpha}_2 = 0.5$, $\overline{\alpha}_3 \approx 0.8184$, and $\overline{\alpha}_4 \approx 0.7722$.

Outline

DE Scaling Analysis for the q-SC with LM1

CS via LM1 DE

LM1 decoding succeeds w.h.p. when error probability (of the q-SC) $\delta \leq \frac{\bar{\alpha}_j}{(k-1)^{j/(j-1)}}$ with $j \geq 2$. In CS language, randomized reconstruction with LM1 succeeds when sparsity $n\delta \leq \frac{n\bar{\alpha}_j}{(k-1)^{j/(j-1)}}$.

Remark

CS by LM1 with randomized reconstruction has an oversampling ratio of $e\lceil \ln \frac{1}{\delta} \rceil$.

DE Scaling Analysis for the q-SC with LM2

CS via LM2 DE

LM2 decoding succeeds w.h.p. when error probability (of the q-SC) $\delta \leq \frac{j}{6k}$ with $j \geq 3$. In CS, randomized reconstruction with LM2 succeeds when $n\delta \leq \frac{nj}{6k}$.

Remark

CS using LM2 with randomized reconstruction has a constant oversampling ratio of 6.

Outline

Scaling Law Analysis for Stopping Sets on the BEC

- Stopping set is the set of nodes that decoding algorithm stops making progress.
- No stopping sets of size less than $n\alpha$ implies decoding succeeds with certainty when fraction of erasure $\delta \leq \alpha$.
- Based on the work on stopping sets analysis on BEC in [ZVO05], we can show that there is a (j, k) LDPC code s.t. decoding succeeds when $\delta < e(k-1)^{-j/(j-2)}$.
- BEC model can not be applied to CS system.

Outline

Scaling Law Analysis for LM1 Stopping Sets

CS via LM1 SS

There is a (j, k)-regular LDPC code ($j \geq 3$) and a constant K such that for all $k \geq K$ all q-SC error patterns of size $n\delta$ for $\delta < e(k-1)^{-j/(j-2)}$ can be recovered by LM1 (w.h.p as $n \to \infty$).

- Idea of the proof:
  - Combinatorial analysis
  - Handle small stopping sets separately

Example

CS with LM1 achieves uniform-in-probability reconstruction by using a (j, k) code with an oversampling ratio of $je^{-(j-2)/j}\delta^{-2/j}$.

Summary and Open Problems

- Connections between coding and compressed sensing
  - Apply verification based decoding alg's to compressed sensing.
- Analyze the scaling law of the MP decoding in the high rate regime for both the BEC and q-SC with LM1/2 decoding)
  - LM2 achieves a constant oversampling ratio for randomized reconstruction
- Stopping set analysis for the BEC and q-SC (with LM1)
- Open Questions
  - Extension of this analysis to stopping set analysis with LM2 (Does it achieve linear scaling?)

Thank you

Thank you