Joint Iterative Decoding of LDPC Codes and Channels with Memory

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Outline

- Channels with Memory
  - APP detectors, the symmetric information rate (SIR), and generalized erasure channels (GECs)
- Low Density Parity Check (LDPC) Codes
  - Density evolution (DE), degree distributions (DDs), and capacity achieving sequences
- Joint Iterative Decoding (JID)
  - A closed form DE recursion for GECs and SIR achieving sequences

Channels with Memory (1)

- Input alphabet $\{0,1\}$ and output alphabet Y
  - Input Seq. = $X_1, \ldots, X_n$ and Output Seq. = $Y_1, \ldots, Y_n$
- Channel Law
  - $W_n(y_1, \ldots, y_n | x_1, \ldots, x_n) = Pr(Y_1, \ldots, Y_n | X_1, \ldots, X_n)$
- Symmetric Information Rate
  - Max rate achievable with i.i.d. equiprobable inputs
  - $I_s = \lim_{n!1} (1/n)\, I(X_1, \ldots, X_n; Y_1, \ldots, Y_n)$

Channels with Memory (2)

- Channel (or APP) Detector
  - Generates log-likelihood ratios (LLRs)

$$L_i = \log \frac{Pr(X_i{=}0|Y_1{=}y_1,\ldots,Y_n{=}y_n)}{Pr(X_i{=}1|Y_1{=}y_1,\ldots,Y_n{=}y_n)}$$

  - Side information via *a priori* probs $p_j = Pr(X_j{=}1)$

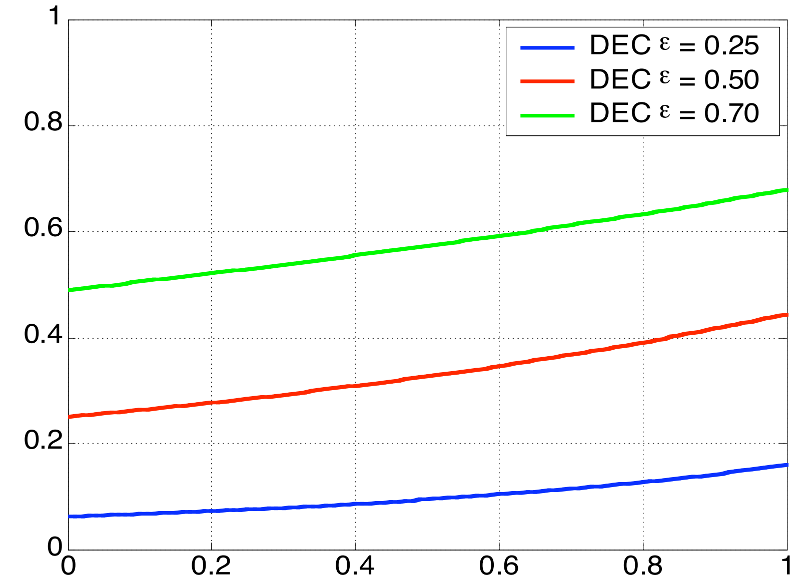
- Example: Finite State Channels
  - Channel law is only a function of channel state
  - Channel detector is given by the BCJR algorithm

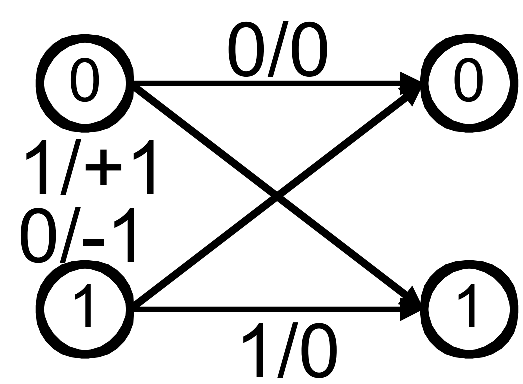
Generalized Erasure Channels

- Symmetric Erasure Distribution (SED)
  - LLR dist. with support $\{-1,0,1\}$ and $Pr(L=1) = Pr(L=-1)$
- Generalized Erasure Channel (GEC)
  - A binary-input channel whose LLR distribution at the channel detector output is an SED whenever the *a priori* LLR distribution is an SED
- Erasure Transfer Function (ETF)
  - Maps the erasure rate of the *a priori* bits, x, to the erasure rate of bits at the detector output, $f(x)$
  - Related to EXIT function by: $T(I) = 1 - f(1-I)$
  - ten Brink's Area Theorem → $I_s = 1 - \int_0^1 f(x)\,dx$

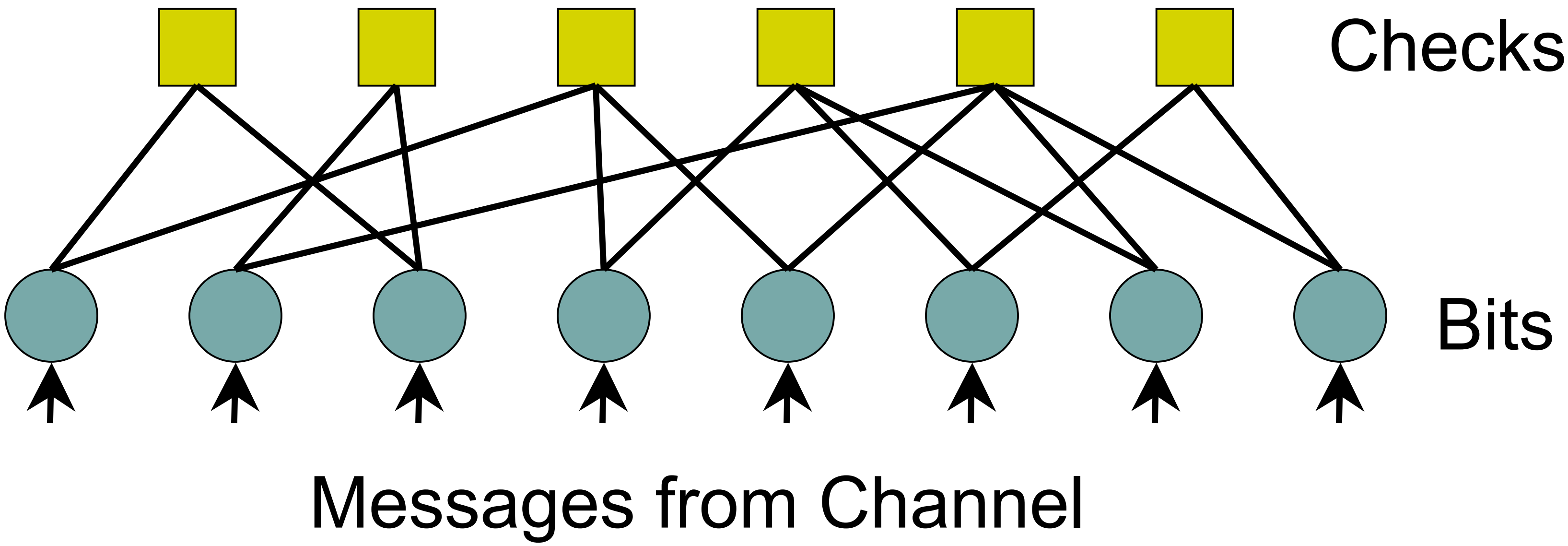
Dicode Erasure Channel (DEC)

- Binary-input linear filter channel with $H(z) = 1\text{-}z$
  - Output erased with prob ε
  - Input $= X_k$
  - Output $= X_k - X_{k\text{-}1}$ (prob $1\text{-}\varepsilon$)
    $=$? (prob ε)
  - SIR: $I_s(\varepsilon) = 1 - 2\varepsilon^2/(1+\varepsilon)$
    - Via mutual information
  - ETF: $f(x) = 4\varepsilon^2/(2\text{-}x(1\text{-}\varepsilon))^2$
    - Via BCJR analysis





LDPC Codes

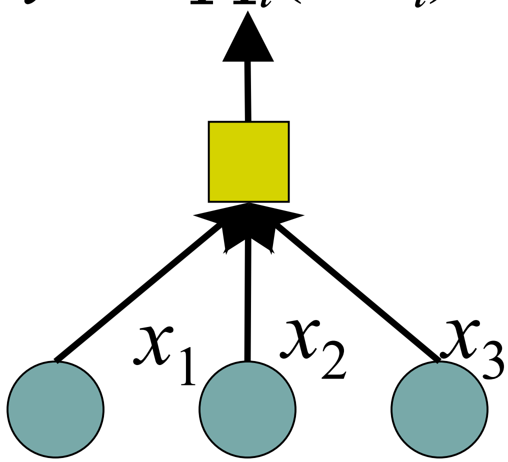


(A) Code's parity check matrix defines the bipartite graph
(B) Decoder passes messages along the edges of the graph
(C) In this talk, all messages will be either 0, 1, or ?
(D) The *degree* of a node is the number edges attached to it

Density Evolution (DE)

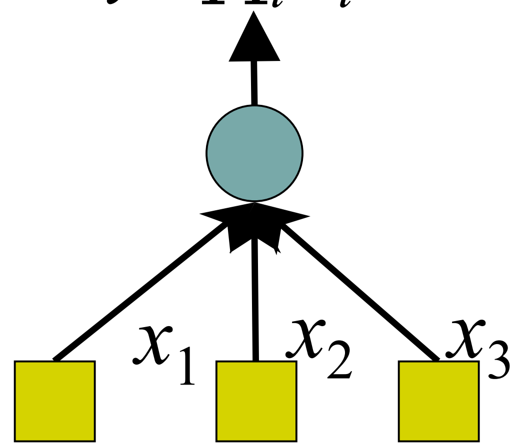
Check output erased if any input erased

$$y=1-\prod_i(1-x_i)$$



Bit output erased only if all inputs erased

$$y=\prod_i x_i$$

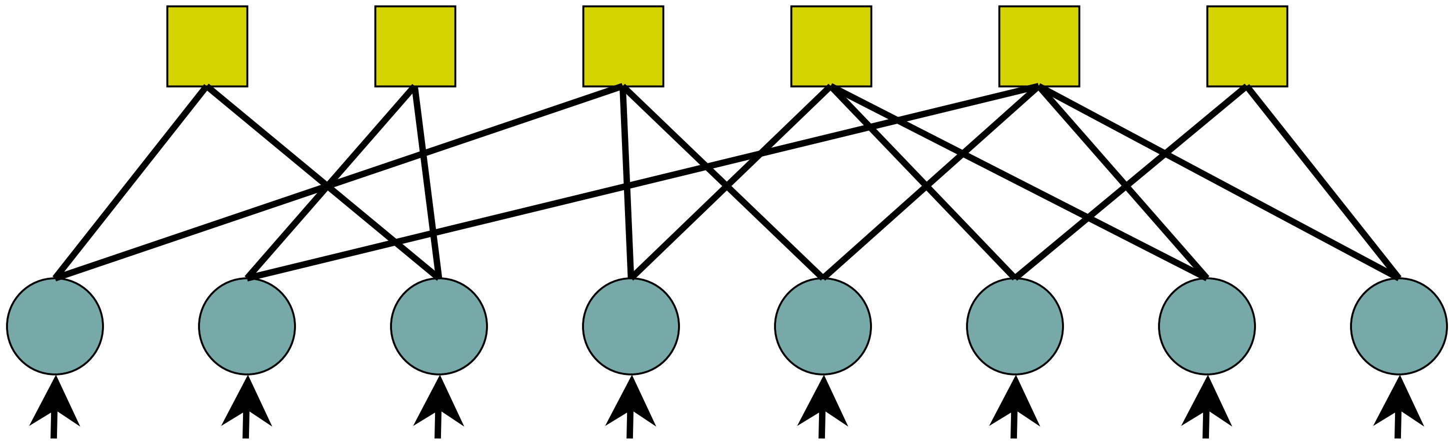


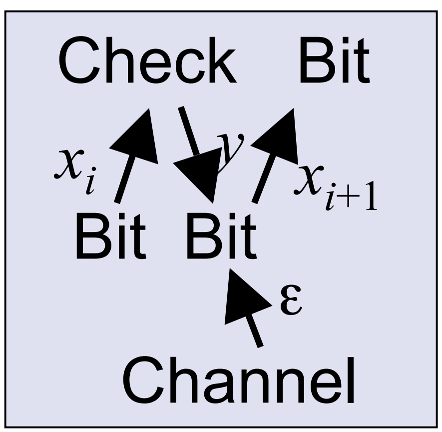
x_i, y = erasure message probability

Degree Distributions (DDs)

- Viewed from either node or edge perspective
  - λ_i (ρ_i) = fraction of edges tied to degree i bit (check) node
  - L_i (R_i) = fraction of bit (check) nodes with degree i
- Written in polynomial form
  - $\lambda(x) = \sum \lambda_i x^{i-1}$ and $\rho(x) = \sum \rho_i x^{i-1}$
  - $L(x) = \sum L_i x^i$ and $\mathrm{R(x)} = \sum R_i x^i$
- Key property (based on counting edges)
  - $\lambda(x) = L'(x)/L'(1)$ and $\rho(x) = R'(x)/R'(1)$
- Code Rate
  - $r = 1 - L'(1)/R'(1) = 1 - (\int_0^1 \rho(x)\, dx \,/\, \int_0^1 \lambda(x)\, dx)$

Density Evolution (DE)



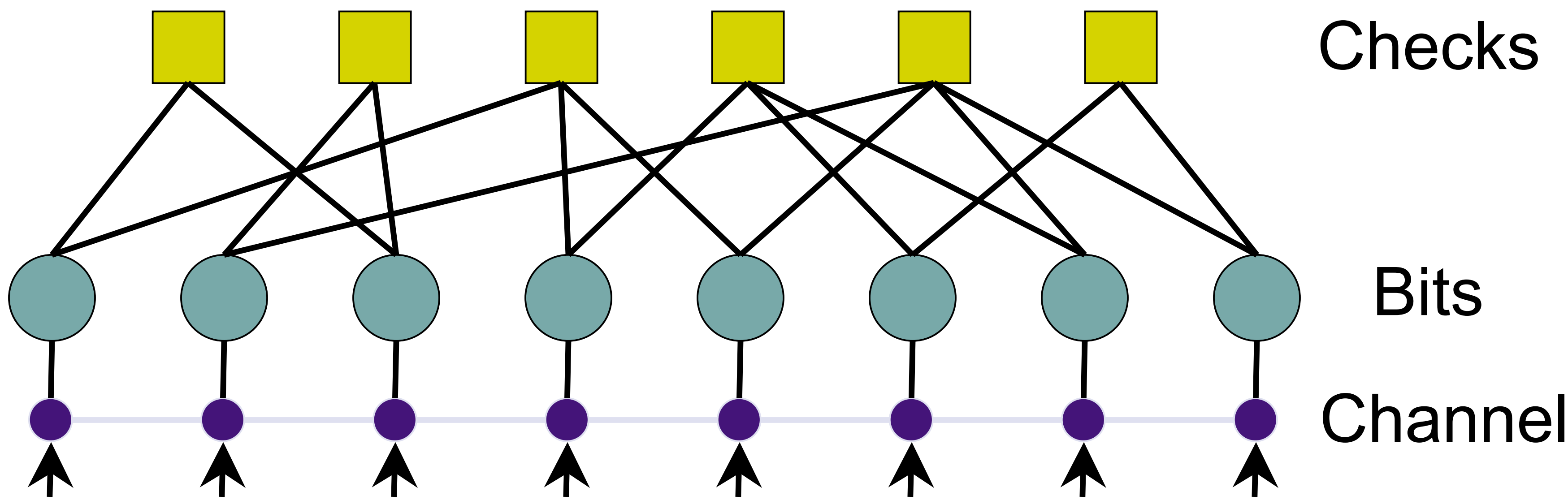


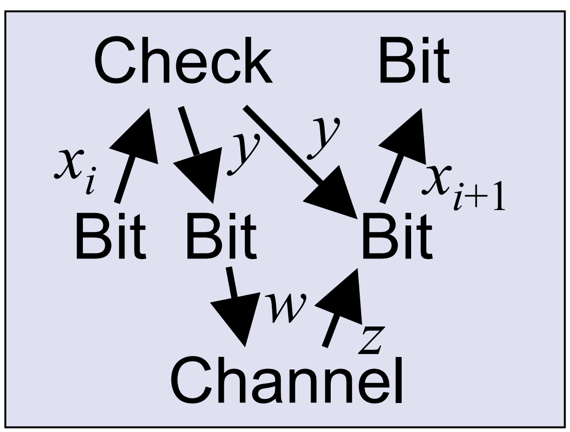
$$y = 1\text{-}\rho(1\text{-}x_i)$$
$$x_{i+1} = \varepsilon\lambda(1\text{-}\rho(1\text{-}x_i))$$

Capacity Achieving Sequences

- DE convergence condition: $\varepsilon\lambda(1-\rho(1-x))<x$
  - Can be rewritten as: $\lambda(x) < (1-\rho^{-1}(1-x))/\varepsilon$
- Regular check DD sequence: $\rho_k(x) = x^{k-1}$
  - Implies bit DD sequence: $\lambda_k(x) < (1-(1-x)^{1/(k-1)})/\varepsilon$
- Expand RHS with a Taylor series about 0
  - All positive coef, but $\lambda_k(1)=1/\varepsilon$ and we need $\lambda_k(1)=1$
  - So we let $\lambda_k(x)$ be the truncated series s.t. $\lambda_k(1)=1$
    - Drops only positive terms → inequality satisfied
  - Code Rate: $r_k \rightarrow (1-\varepsilon)$ as $k \rightarrow \infty$ (capacity achieving)

Joint Iterative Decoding (JID)





$y = 1\text{-}\rho(1\text{-}x_i)$

$w = L(y)$ (one msg per bit)

$z = f(w)$ (channel ETF)

$x_{i+1} = f(L(1\text{-}\rho(1\text{-}x_i)))\lambda(1\text{-}\rho(1\text{-}x_i))$

SIR Achieving Sequences (1)

- DE convergence condition for JID
  - $f(L(1-\rho(1-x)))\lambda(1-\rho(1-x))<x \rightarrow f(L(x))\lambda(x)<1-\rho^{-1}(1-x)$
    - Define $q(x) = 1-\rho^{-1}(1-x)$
- If inequality holds and $\lambda(x) \rightarrow L'(x)/L'(1)$
  - Can rearrange to $g(x)=L'(1)q(x)-f(L(x))L'(x) > 0$
- Rate Gap Theorem
  - Gap to SIR $= I_s - r = \int_0^1 g(x)dx > 0$
  - Must have $g(x) = 0$ (a.e.) to achieve the SIR

SIR Achieving Sequences (2)

- Achieving the SIR requires equality and this gives the following differential equation
  - $F(x) = \int_0^x f(t)dt$ and $Q(x) = \int_0^x q(t)dt = \int_0^x 1-\rho^{-1}(1-t)dt$

$$
\begin{aligned}
f(L(x))\lambda(x) &= q(x) \\
f(L(x))L'(x) &= L'(1)q(x) && (1) \\
F(L(x)) &= L'(1)Q(x) && (2) \\
L(x) &= F^{-1}(L'(1)Q(x)) && (3) \\
\lambda(x) &= q(x)/f(F^{-1}(L'(1)Q(x)) && (4)
\end{aligned}
$$

(1) $\lambda(x) \rightarrow L'(x)/L'(1)$ (2) $\int_0^x$ (3) Solve for $L(x)$ (4) $\frac{1}{L'(1)}\frac{d}{dx}$

SIR Achieving Sequences (3)

- Unlike BEC, truncation violates the inequality
  - Instead, use a sequence of degraded channels
    - $f^{(k)}(x) = (1+\alpha_k)\, f(x)$
  - Pick check DD seq. $\rho^{(k)}(x)$ and α_k seq. such that
    - $f^{(k)}(x)(L^{(k)}(1-\rho^{(k)}(1-x)))\lambda^{(k)}(1-\rho^{(k)}(1-x)) < x$ (convergence)

$$\lambda^{(k)}(x) = \frac{(1+\alpha_k)^{-1} q^{(k)}(x)}{f(F^{-1}(F(1) Q^{(k)}(x)/Q^{(k)}(1)))}$$

- SIR Achieving Sequences Theorem
  - If $\lambda^{(k)}(x)$ well-behaved then $\alpha_k \to 0$ and $r_k \to I_s$

Results

- Used method to design codes for the DEC
  - Check regular sequences: $\rho_k(x) = x^{k\text{-}1}$
- DEC with ε = 0.85
  - $k = 5 \rightarrow \lambda^{(k)}(x)$ truncation to 757 terms ($\lambda^{(k)}(1) = 1$)
  - I_s = 0.2189, Rate gap = 0.0002, and $\alpha_k = 0.00003$
- Precoded DEC with ε = 0.5
  - $k = 11 \rightarrow \lambda^{(k)}(x)$ truncation to 184 terms ($\lambda^{(k)}(1) = 1$)
  - $I_s = 2/3$, Rate gap = 0.0009, and $\alpha_k = 0.0023$

Conclusions

- Introduced algebraically defined DD seq. for a class of erasure channels with memory
  - Can show that these sequences achieve the SIR under certain conditions
  - Applied this method to design codes for the DEC
    - We conjecture that the code seq. approaches the SIR
    - Gap in the proof: $\lambda^{(k)}(x)$ has all positive coefficients?
  - Open problem
    - For any $f(x)$, does there exist a $\rho^{(k)}(x)$ sequence such that this construction achieves the SIR?