Coding Theorems for Generalized Repeat Accumulate Codes

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ISITA 2000, Honolulu, Hawaii
November 5-8, 2000

Outline

- Generalized Repeat Accumulate (GRA^m) codes
- The coding theorem
- Iterative decoding of GRA^m codes
- Conclusions

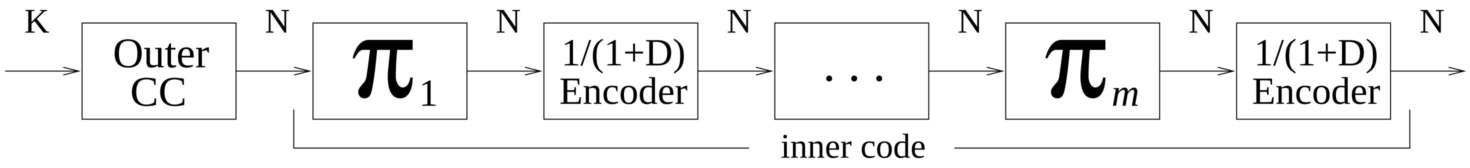
Background

- Turbo Codes (Berrou *et al.*)
  - Analysis uses the Uniform Random Interleaver (Benedetto *et al.*)
  - Iterative decoding gives exceptional performance at low SNR
- Repeat Accumulate (RA) Codes (Divsalar *et al.*)
  - A repetition outer code, a random permutation, and a rate-1 code
    - Proven to have vanishing word error if repeat order $q \geq 3$
  - Outer terminated convolutional codes -> $d_{free} \geq 3$ (Jin *et al.*)
- Union Bound (UB) (upper bound on word error probability)

$$P_W(N) \leq \sum_{h=1}^{N} A_h(N) z^h$$

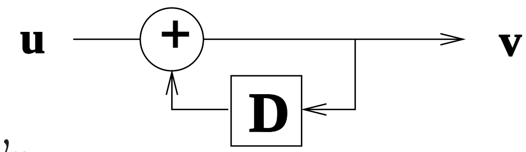
  - Chernoff bound of pairwise error probability-> parameter z
  - Bit error rate bounded by using $B_h(N) = \sum_{w=1}^{RN} \frac{w}{RN} A_{w,h}(N)$

Generalized Repeat Accumulate Codes



- Generalized Repeated Accumulated (GRA^m) Codes
  - The outer code is a terminated convolutional code
    - Includes repeated block codes as the class of 0 memory CCs
  - Inner code is the SC of m rate-1 "Accumulate" Codes
- Generalizes RA Codes in two ways
  - Uses terminated convolutional code as outer code (Jin *et al.*)
  - Allows for more than one "accumulate" code (Pfister *et al.*)
- Motivated by the analysis of GRA^m codes for large m (Pfister *et al.*)
  - At any N, most of the codes have d_{min} which approaches the GV Bound

The Rate-1 Accumulate Code



- Accumulate over block:

$$u_1 u_2 \ldots u_n \;\rightarrow\; v_1 v_2 \ldots v_n$$
$$v_1 = u_1$$
$$v_2 = u_1 + u_2$$
$$\vdots$$
$$v_n = u_1 + \ldots + u_n$$

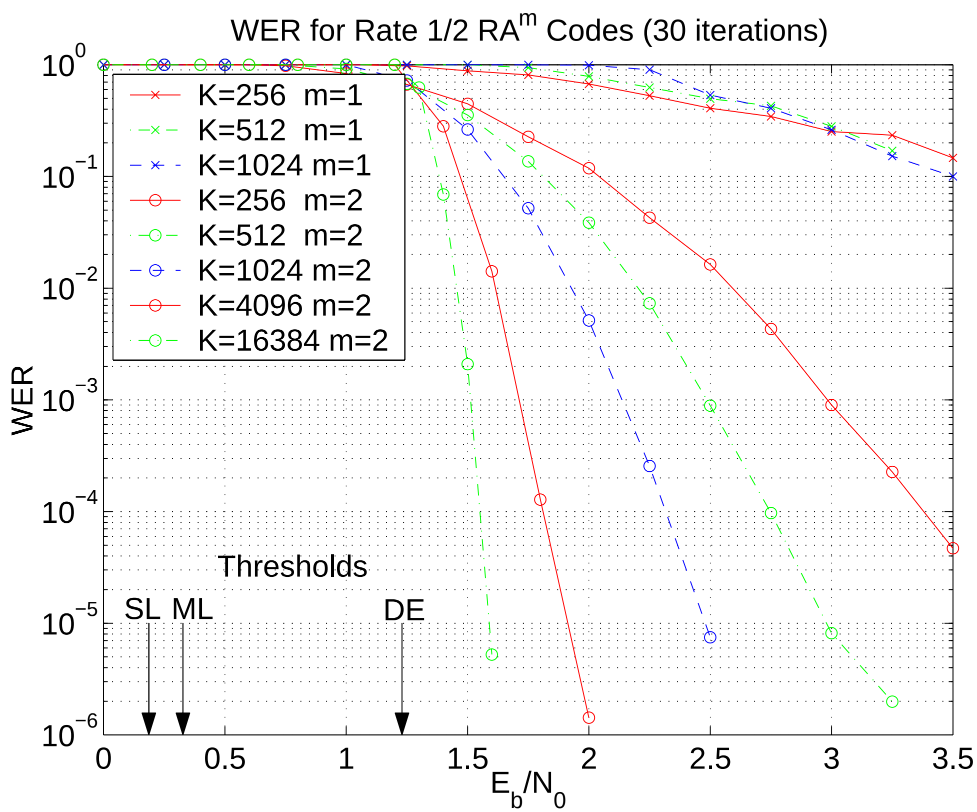
- One important property: The Hamming weight can't decrease "too" much
  - Write the inputs in the form $0^{b_1}\underline{10^{a_1}1}0^{b_2}\underline{10^{a_2}1}\ldots 0^{b_k}\underline{10^{a_k}1}0^{b_{k+1}}$
  - Each weight 2 input fragment produces exactly $a_i + 1$ ones at the output
    - For odd input weights, the last "1" produces at least one "1"
  - For input weight w and output weight h, we have $h \geq \lceil w/2 \rceil$
    - For fixed w and large N, prob. of equality $\approx \binom{N}{\lfloor w/2 \rfloor} / \binom{N}{w} = O(N^{-\lceil w/2 \rceil})$

A Coding Theorem for GRA^m Codes

- **Assumptions:**
  - ML decoding of GRA^m code ensemble with outer CC $d_{free} \geq 2$
  - Pairwise error probability upper bounded by z^h
- **Theorem:** There exists a positive threshold z^* such that, for any $z < z^*$
  - The average probability of word error is $P_W = \Theta(N^\nu)$
    - where the decay rate $\nu = 1 - \sum_{i=1}^{m} \lceil d_{free}/2^i \rceil$
  - The average probability of bit error is $P_B = \Theta(N^{\nu-1})$
    - If the outer code satisfies $w \leq th$ for some constant t
- **New Results:** If $d_{free} = 2$ and $m = 1$ we have $P_B \to 0$ as $N \to \infty$
  If $d_{free} = 2$ and $m > 1$ we have $P_W \to 0$ as $N \to \infty$
- **Note:** $f(n) = \Theta(g(n))$ means that $\exists n_0, c_1, c_2$ such that
  $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n > n_0$

Iterative Decoding of GRA^m Codes (RA^1 vs. RA^2)

- Word error probability via simulation (SL=0.187 dB, ML=0.327 dB, DE=1.23 dB)



Bounding the Probability of Error

- Upper bound the average WE $\overline{A}_{h_{m+1}}$ using

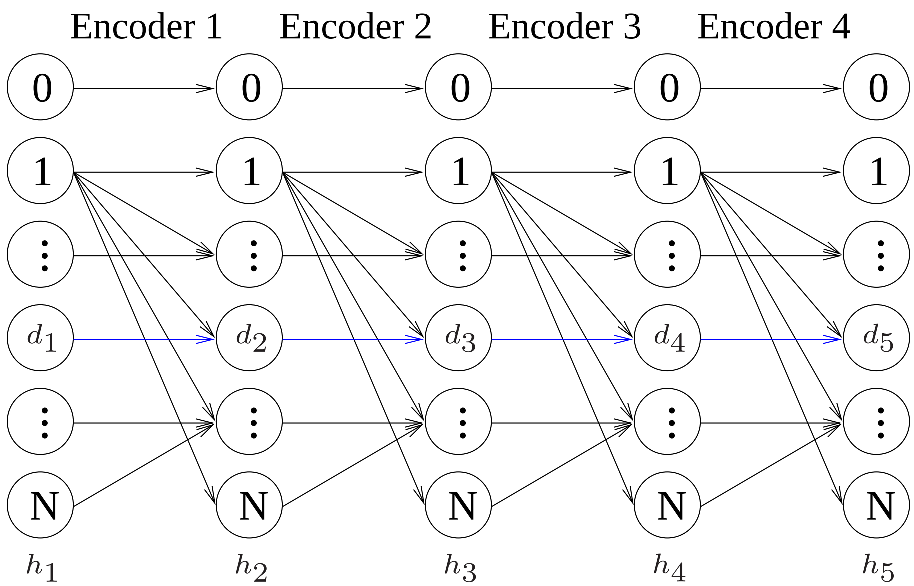
$$\overline{A}_{h_{m+1}}(N) = \sum_{(h_1,\ldots h_m)=(1,\ldots,1)}^{(N,\ldots,N)} A^{(o)}_{h_1}(N) \prod_{i=1}^{m} \frac{A^{(acc)}_{h_i,h_{i+1}}(N)}{\binom{N}{h_i}}$$

  where $A^{(o)}_{h_1}$ is the outer code WE and $A^{(acc)}_{h_i,h_{i+1}}$ is the "accumulate" IOWE

- Union Bound the ensemble averaged probability of error
  - Sum separately large and small ($h_{m+1} \gtrless h_N$) weights (Divsalar *et al.*)
    - Boundary weight h_N is chosen to grow slowly s.t. $\ln h_N / \ln N = o(1)$
    - Large output weights sum determines the noise threshold z^*
    - Small output weights sum determines rate of error decay ν
    - Min. distance "weight path" ($h_i = \lceil d_{free}/2^{i-1} \rceil$) dominates

"Weight Paths" through the Encoders

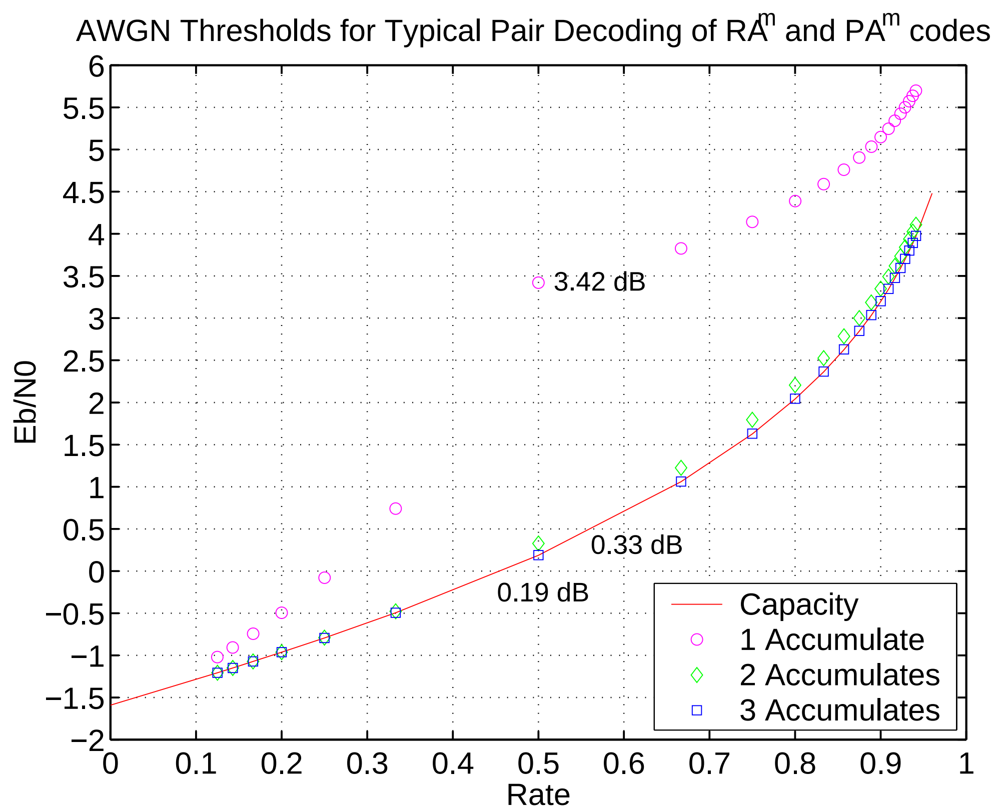
- Each "weight path" $h_1, \ldots, h_{m+1}$ can be analyzed separately



- The minimum distance "weight path" has $h_i = d_i = \lceil d_{free}/2^{i-1} \rceil$
  - Path takes the max wt. drop at each step -> probability $= O(N^{\nu-1})$
  - With $O(N)$ inputs of weight d_{free}, this lower bounds the WER at $O(N^{\nu})$

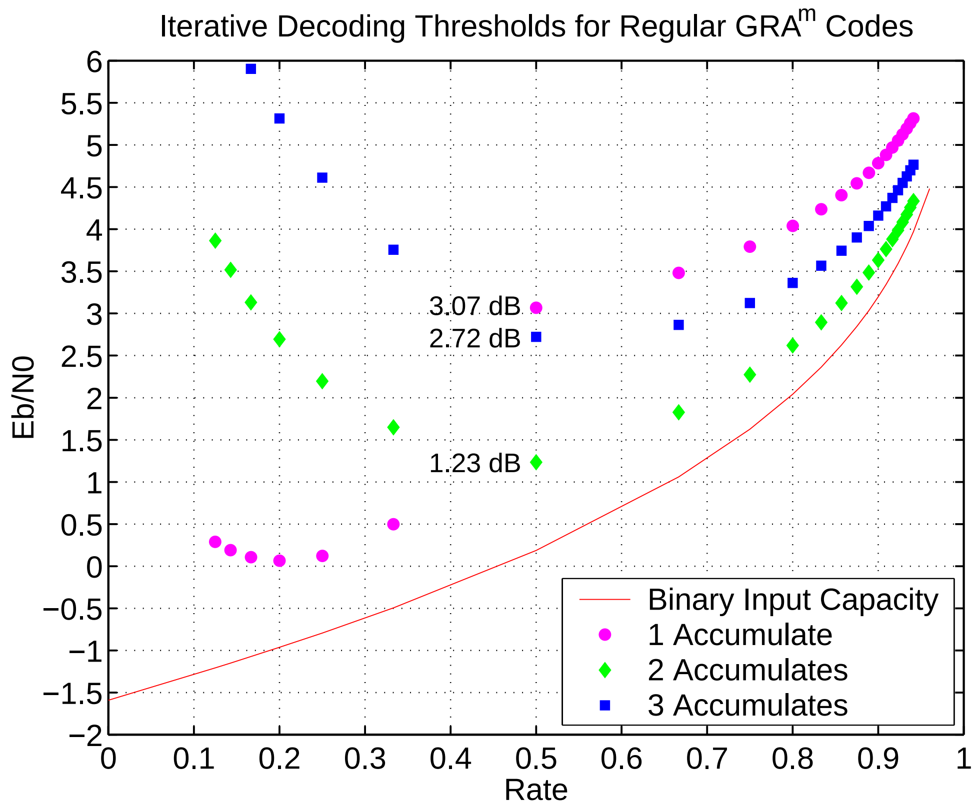
$\mathrm{E_b/N_0}$ Thresholds for ML Decoding

- Using the new Joint Typicality (JT) bound for error probability (Jin *et al.*)



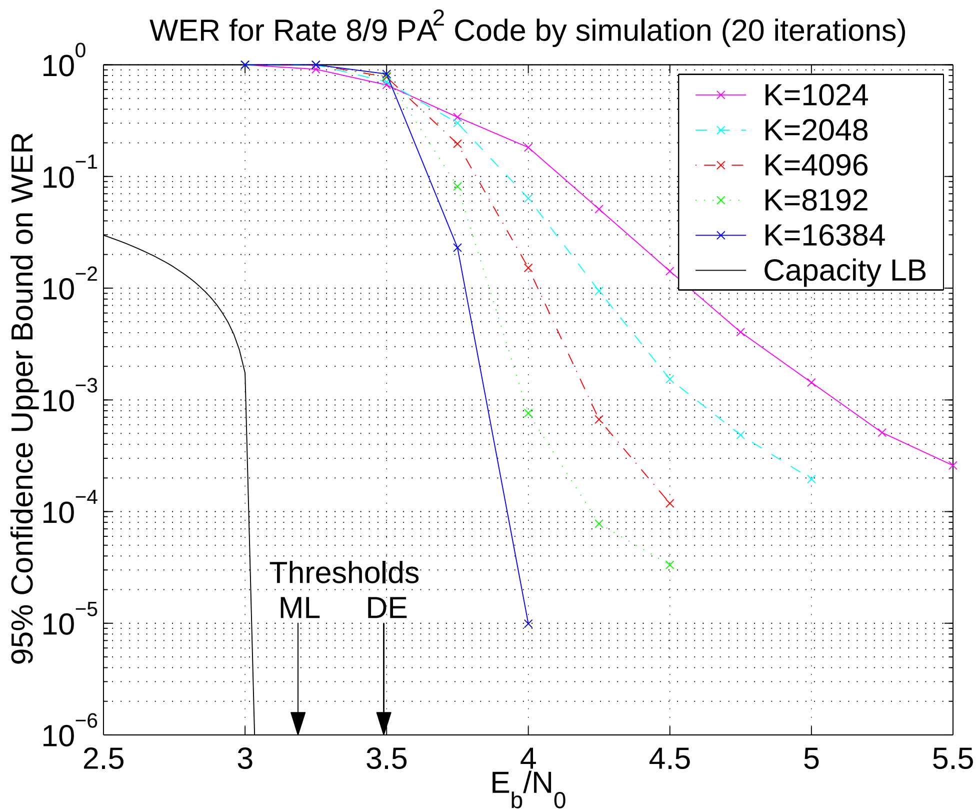
E_b/N_0 Thresholds for Iterative Decoding

❍ Applying density evolution to GRA^m codes (message passing decoder)



PA^2 Codes for High Rate Applications

- High rate PA^2 codes perform well (SL=3.033 dB, ML=3.187 dB, DE=3.49 dB)



Conclusions

- Coding Theorem
  - Extends the work of (Divsalar *et al.*) and (Jin *et al.*) to $d_{free} = 2$ and $m > 1$
- Simulation and Threshold Analysis
  - ML decoding analysis gives extremely good noise thresholds for $m \geq 3$
  - Density evolution gives good noise thresholds for high rate $m = 2$
    - The thresholds are very similar to regular high rate LDPC codes
- Open Questions
  - What is the typical behavior of GRA^m codes under ML decoding?
    - Average behavior is quite pessimistic, dominated by bad codes