Finite-Length Analysis of a Capacity-Achieving Ensemble for the Binary Erasure Channel

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Outline

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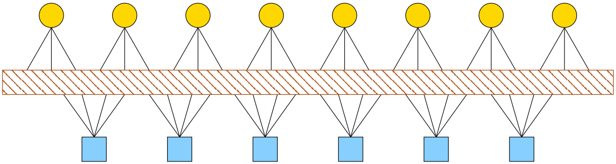
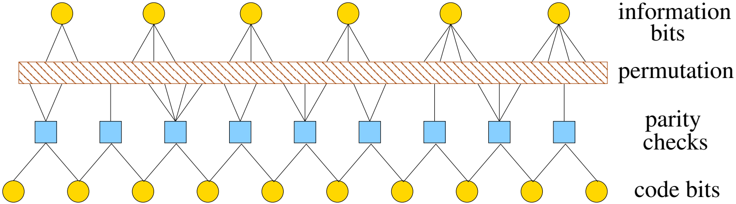
Low Density Parity Check (LDPC) Codes



- Linear codes with sparse parity-check matrix H
  - Regular (j, k): H has j ones per column and k ones per row
  - Irregular (λ, ρ): uses degree distributions for ones in H
- Bipartite Graph
  - An edge connects check node i to bit node j if $H_{ij} = 1$
  - Used for *message passing iterative* (MPI) decoding

Irregular Repeat-Accumulate (IRA) Codes



- Can be viewed either as a Turbo or LDPC variation
  - LDPC: Simply add zig-zag structured degree 2 bits
  - Turbo: Repeat info bits, parity-check, and accumulate
- Repeat-parity given by sparse generator matrix G
  - Information bit j included in parity check i if $G_{ij} = 1$
  - Regular (j, k): G has j ones per column and k ones per row
  - Irregular (λ, ρ): uses degree distributions for ones in G

Degree Distributions and Density Evolution

- Definition: *degree distribution* (d.d) function

$$\lambda(x) = \sum_{i\geq 1} \lambda_i x^{i-1} \qquad \rho(x) = \sum_{i\geq 1} \rho_i x^{i-1}$$

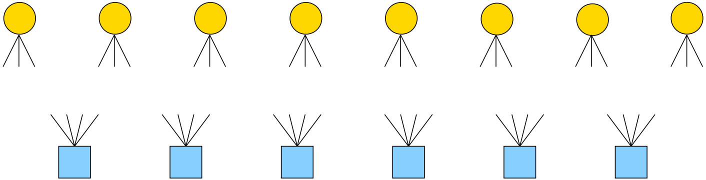
  - λ_i = Fraction of edges attached to bits of degree i
  - ρ_i = Fraction of edges attached to checks of degree i

- Density evolution (DE)
  - Tracks distribution of messages during iterative decoding
  - Long codes decode almost surely if DE converges
  - For BEC, let x_i = erasure rate of bit output messages

$$x_{i+1} = p\lambda\left(1 - \rho(1 - x_i)\right)$$

Outline

Peeling Style Analysis of LDPC Codes (Luby et al.)



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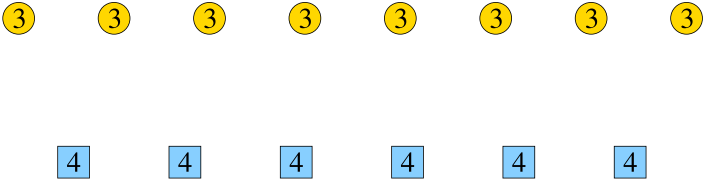

$$\{0, 0, 0, 24\}$$

Number of Edges by Degree

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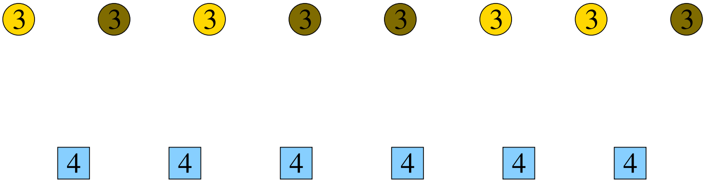

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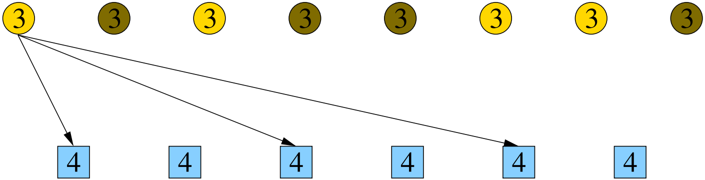

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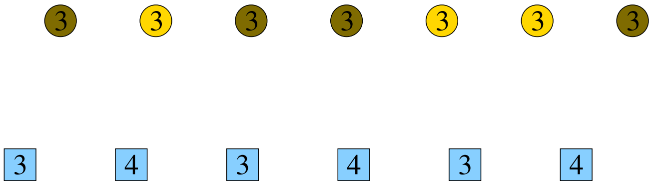

$$\{0, 0, 9, 12\}$$

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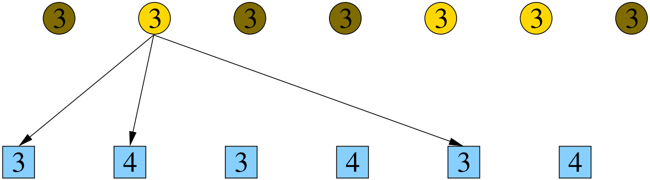

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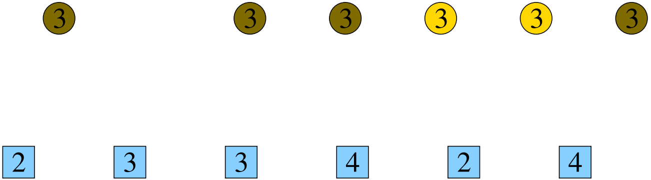

$$\{0, 4, 6, 8\}$$

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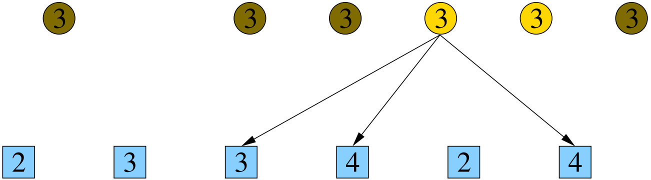

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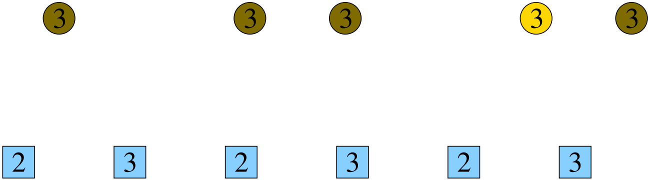

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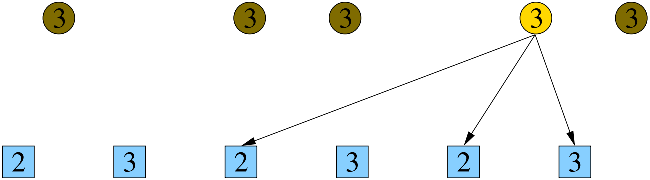

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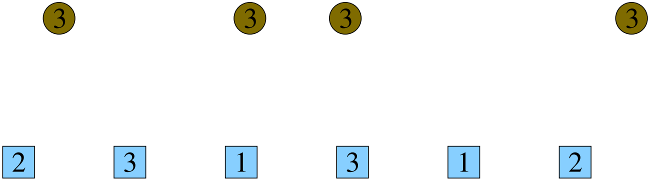

$$\{2, 4, 6, 0\}$$

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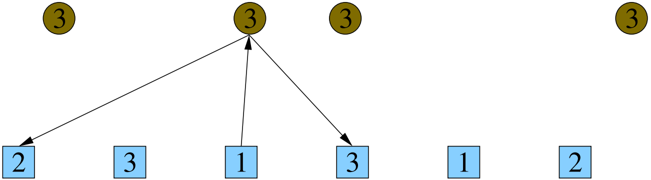

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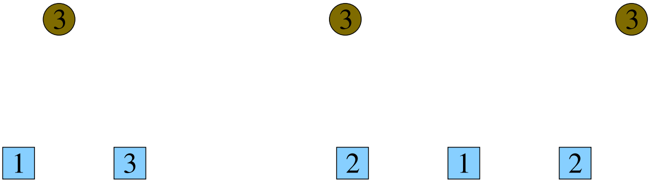

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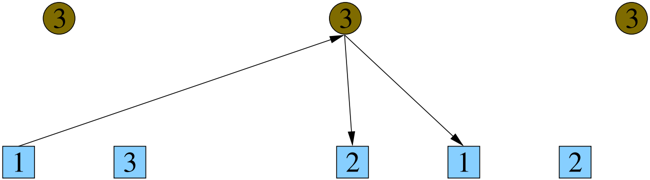

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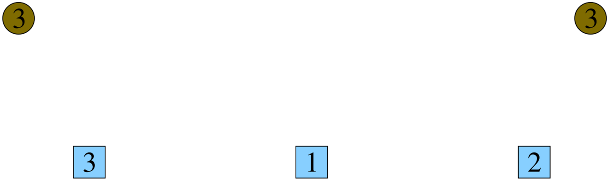

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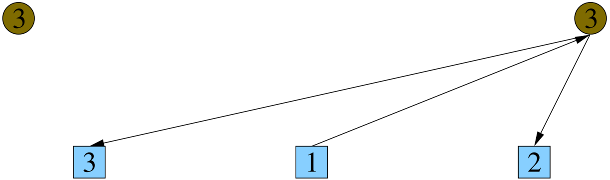

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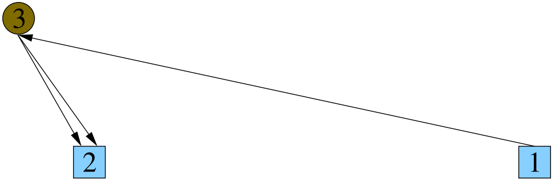

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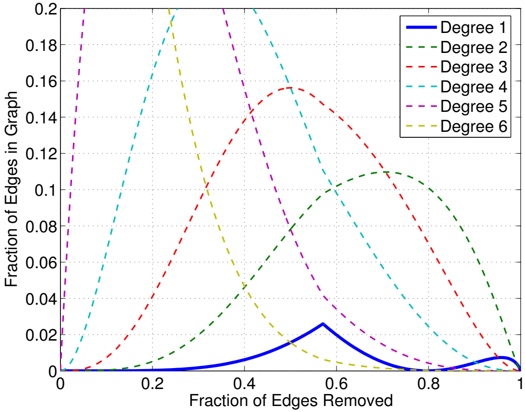
Number of Edges by Degree

Peeling Style Analysis of LDPC Codes (Luby et al.)

Decoding Successful

Mean Trajectory for the (3,6) LDPC Code



- *Critical point* is where the fraction of deg. 1 edges is zero

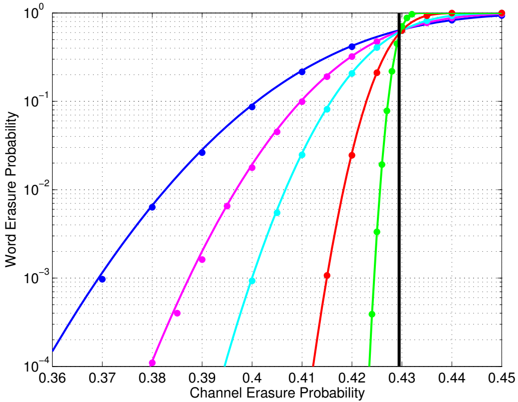
Finite Length Scaling for LDPC Codes

- Refined analysis of peeling style decoding (Amraoui et al.)
  - Number of bit and check edges asymptotically Gaussian
    - Use differential equations to track the mean and covariance
  - Probability of block error versus block length n given by

$$P_B = Q\left(\frac{\sqrt{n}(p^* - \beta n^{-2/3} - p)}{\alpha}\right) + o(1)$$

  - Exact in the limit as $n \to \infty$ with $\sqrt{n}\,(p^* - p)$ held constant
- Parameters defined in the neighborhood of the critical point
  - α related to std. dev. of number of degree 1 edges
  - β related to width of parabola at the critical point

Scaling Results for (3,6) LDPC



- Parameters: $p^* = 0.42944$, $\alpha = 0.56036$, and $\beta = 0.61695$
- Block length: $n = 1024, 2048, 4096, 16384, 131072$
- Outer code assumed to eliminate small stopping sets

Covariance Evolution for LDPC Decoding

- Bit Regular Decoding
  - Assume we start with n check edges
  - Let $X_{n,i}^{(j)}$ be the number of deg. j check edges after i steps
  - Number check edges of each deg. is a Markov process
- Phase 1: Remove $(1-p^*)n$ edges for known bits
  - Pick random edge $\sim X_{n,i}^{(j)}/(n-i)$
  - If deg. k, replace k deg. k edges with $k-1$ deg. $k-1$ edges
  - Differential eq. for mean and covariance (Amraoui et al.)
- Phase 2: Remove $(t_{crit}-1+p^*)n$ edges for decoding
  - Remove a degree 1 edge
  - Repeat $d-1$ times: Remove random edge as above
  - Differential eqns for mean and covariance (Amraoui et al.)
- Parameter α given by the variance of degree 1 edges

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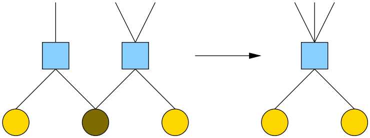
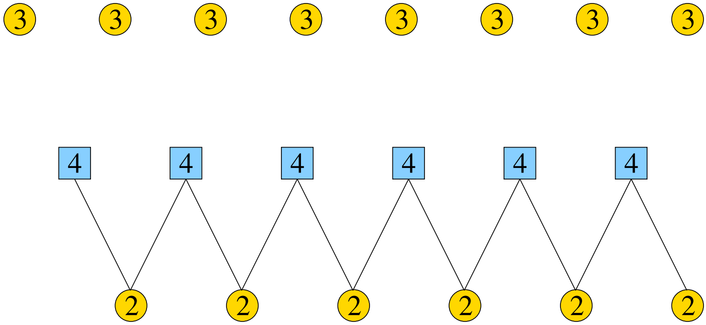
Graph Reduction For IRA Codes



- *Graph reduction* removes all code bits from the graph
  - Peeling style decoding removes all known code bits
  - Merging check nodes removes all erased code bits
    - Equivalent to summing check equations to remove bit
- After graph reduction we have
  - A standard LDPC code with a modified check d.d.
  - Check d.d. is random and depends on erased code bits
- Straightforward generalization of scaling also possible
  - A *degree vector* for each node, but complexity increased

Peeling Style Analysis for IRA Codes



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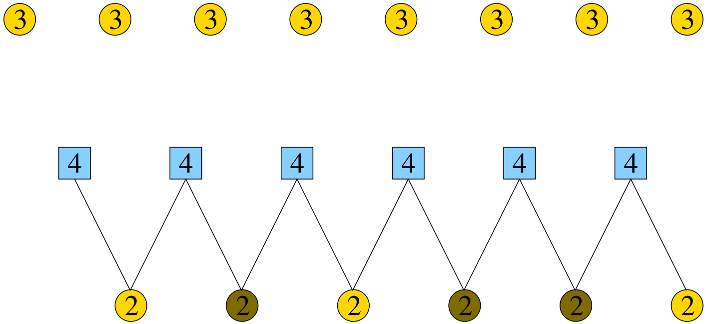

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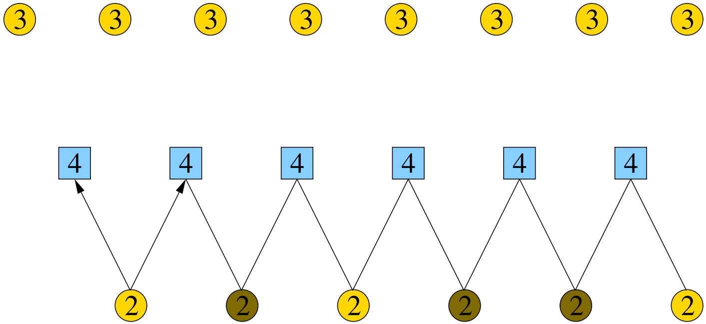

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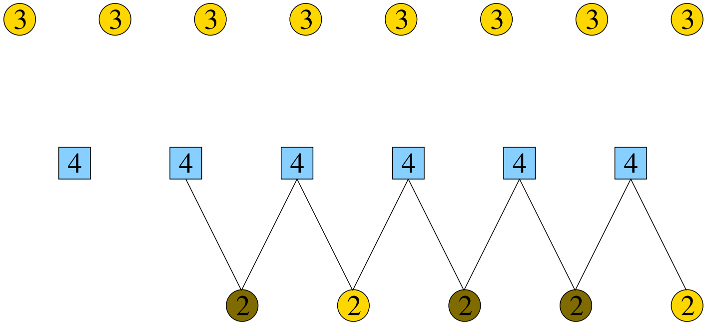

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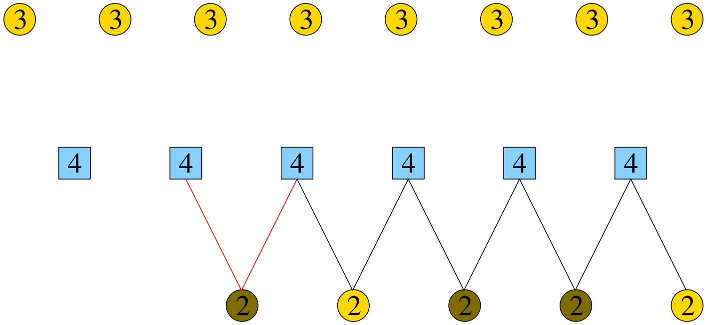

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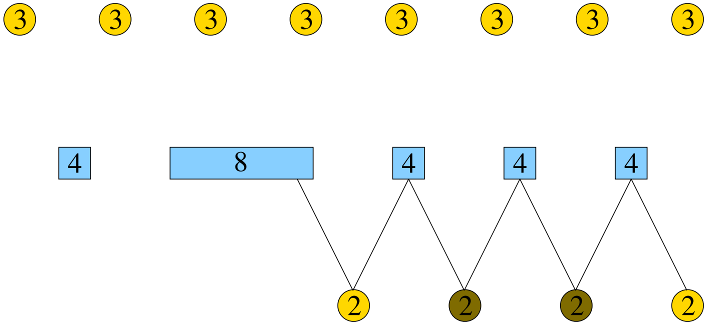

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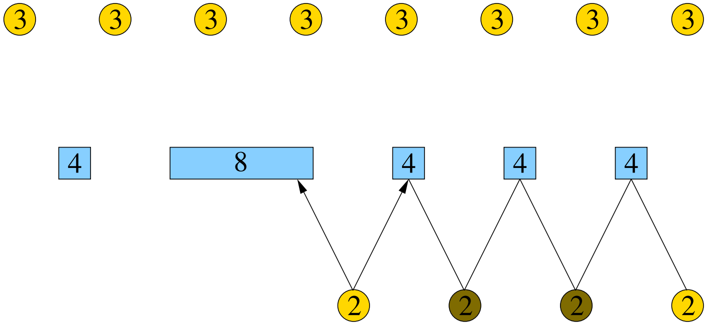

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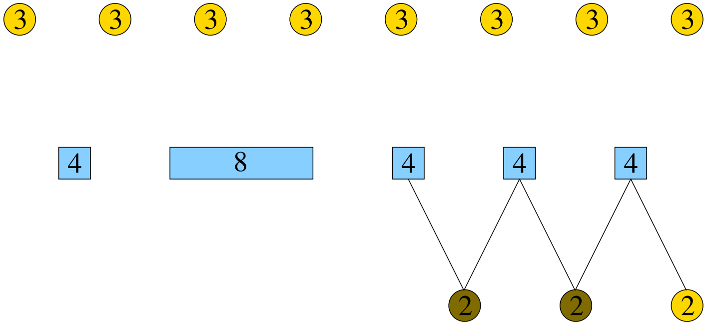

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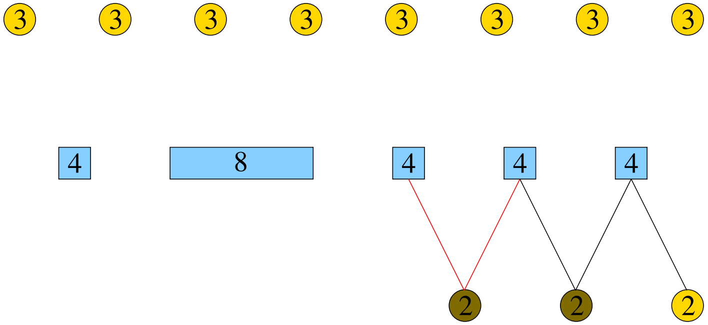

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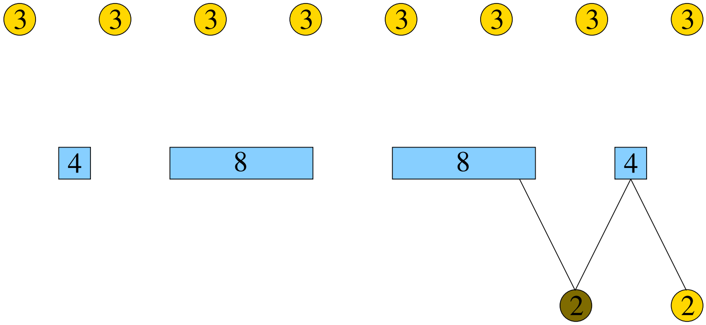

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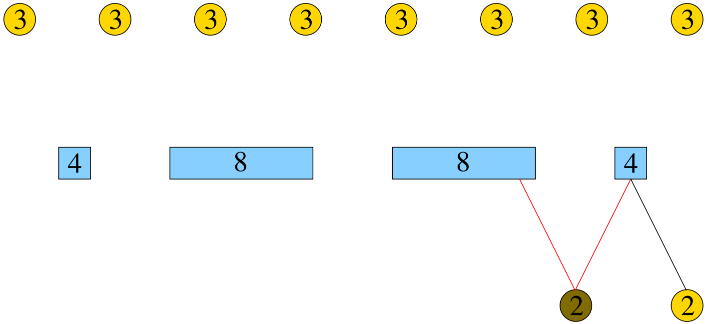

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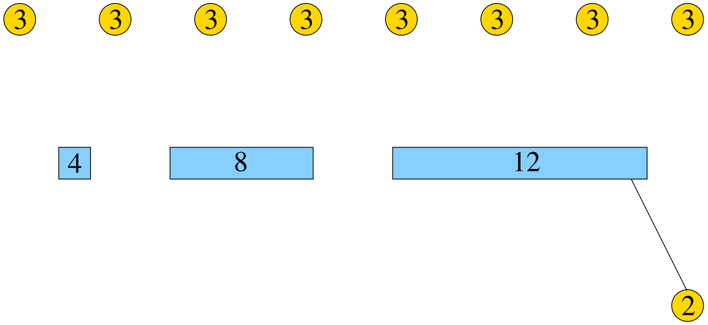

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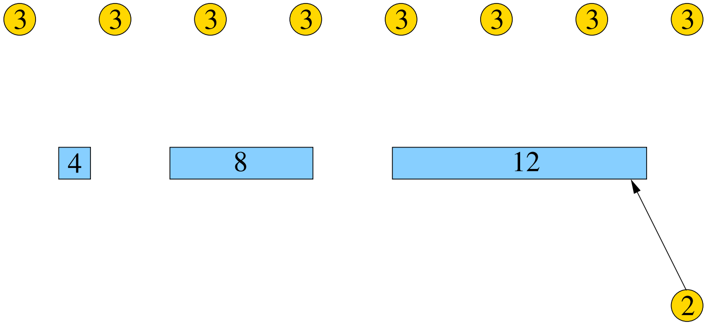

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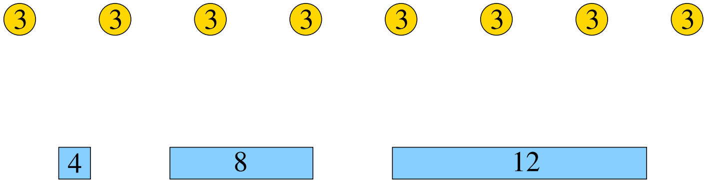

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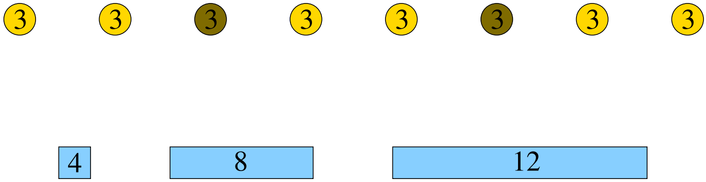

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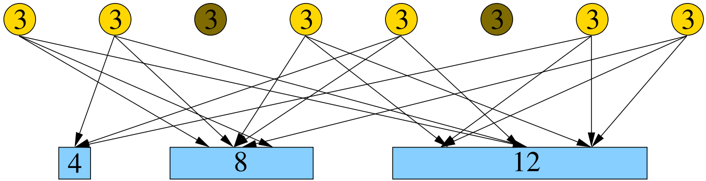

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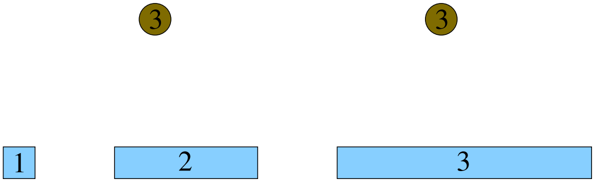

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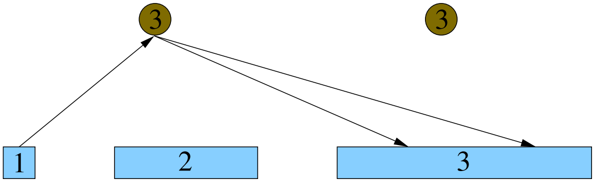

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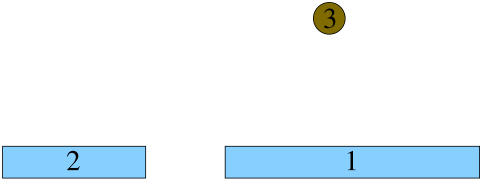

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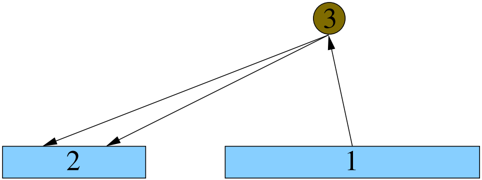
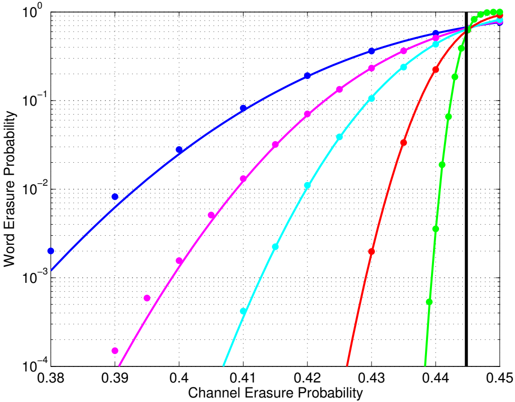


Peeling Style Analysis for IRA Codes

Decoding Successful

Rate 1/2 Systematic (3,3) IRA Code



- Parameters: $p^* = 0.44478$ $\alpha = 0.59588$ $\beta = 0.83874$
- Block length: $n = 1024, 2048, 4096, 16384, 131072$
- Outer code assumed to eliminate small stopping sets

Covariance Evolution for Graph Reduction

- Graph Reduction (starting with n checks)
  - Number check nodes of each deg. is a Markov process
  - State $X_{n,i}^{(j)}$ is number of checks of degree j after i steps
  - $X_{n,0}^{(j)} = n\,R_j$ where R_j is the fraction of check nodes deg. j
  - For each erasure, pick two checks and combine

$$Pr(\deg j,\ \deg k \rightarrow \deg j+k) = \frac{X_{n,i}^{(j)} X_{n,i}^{(k)}}{(n-i)(n-i)} + O\left(\frac{1}{n}\right)$$

  - This is sufficient to apply the theorem of Amraoui et al.
- Conversion to edge perspective (to continue decoding)
  - Number of edges deg. j after i steps: $Y_{n,i}^{(j)} = \frac{j\,X_{n,i}^{(j)}}{\sum_k k\,R_k}$

Outline

Approaching Capacity in Practice

- The biggest obstacle is the enormous block length required
  - Irregular LDPC codes limited by length, not complexity
  - Length 10^7 used for Chung's 0.04 dB from capacity result
- Block length vs. gap to capacity for iterative decoding?
  - First, need a capacity achieving sequence of ensembles
  - Second, need to pick a block length for each ensemble
  - Empirically: If length grows too slowly, performance is bad
- Two Approaches
  - Scaling law: Determine $\{p^*, \alpha, \beta\}$ for c.a. sequence
  - Weight enumerator: Focus on low weight codewords

Capacity-Achieving LDPC Codes for the BEC

- A seq. of codes is *capacity-achieving* (c.a.) on a channel
  - If DE converges for each code in the sequence
  - Sequence of code rates converges to channel capacity
- Complexity of iterative decoding
  - Proportional to number of edges in the graph
- Check regular c.a. sequence $\{\lambda^{(k)}, \rho^{(k)}\}$ (Shokrollahi)
  - Let $\rho^{(k)}(x) = x^k$ and $\widetilde{\lambda}^{(k)}(x) = \frac{1}{p}\left(1-(1-x)^{1/k}\right)$
  - $\lambda^{(k)}(x)$ given by truncating series for $\widetilde{\lambda}^{(k)}(x)$ so $\lambda^{(k)}(1) = 1$
  - Complexity grows like $\ln \frac{1}{\varepsilon}$ for gap to capacity ε

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Capacity-Achieving IRA Codes for the BEC

- Density Evolution (Turbo style decoding):

$$x_{i+1} = \lambda \left(1 - \left(\frac{1-p}{1-pR(1-x_i)}\right)^2 \rho(1-x_i)\right)$$

- Bit regular non-sys. IRA Ensemble $\lambda(x) = x^2$ (deg. 3)

$$\rho(x) = \sum_{i\geq 1} \rho_i x^{i-1} = \frac{1-(1-x)^{1/2}}{\left(1-p\left(1-3x+2\left(1-(1-x)^{3/2}\right)\right)\right)^2}$$

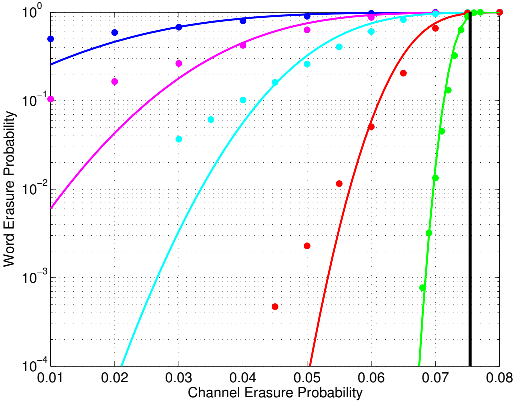
- Sequence of ensembles $\left\{\lambda, \rho^{(M)}\right\}$ by truncation of $\rho(x)$
  - where $\rho_M(x) = \sum_{i=2}^{M-1} \rho_i x^{i-1} + \sum_{i=M}^{\infty} \rho_i x^{M-1}$
  - Capacity achieving for $p \leq 1/13$
  - Complexity converges to $3 + \frac{2}{1-p}$ and $\varepsilon = O\left(M^{-1/2}\right)$

Scaling for Capacity-Achieving IRA Sequence

Code	γ	Rate	p^*	α	β
IRA M=20	.0019	.9126	.0754	.4122	2.938
IRA M=30	.0021	.9173	.0754	.4842	4.079
IRA M=40	.0020	.9194	.0754	.5684	5.462
IRA M=50	.0019	.9206	.0753	.6577	7.017
IRA M=60	.0017	.9214	.0753	.7491	8.737

- Bit regular (degree 3) c.a. non-systematic IRA codes
- Design rate = 0.925, γ = fraction of sys. bits transmitted
- Parameter α rising slowly, but β rising quickly
    - Need bounds $\overline{\alpha}_M, \overline{\beta}_M$ on α, β as a function of M
    - Then, choose n_M so $\overline{\alpha}_M n_M^{-1/2}$ and $\overline{\beta}_M n_M^{-2/3}$ are bounded

Capacity-Achieving IRA Sequence M=40



- Parameters: $p^* = 0.754$ $\alpha = 0.5684$ $\beta = 5.462$
- Block length: $n = 1024, 2048, 4096, 16384, 131072$
- Real problem: Scaling law convergence not uniform

Weight Enumerator (WE) Analysis

- An IRA encoder is the serial concatenation of a
  - Repeat code IOWE: $A^{(rep)}_{p,s} = \binom{nR'(1)/3}{p}\delta_{s,3p}$
  - Parity code IOWE: $A^{(par)}_{s,q} \leq \binom{n}{q}(R'(1))^q \frac{\left(\frac{1}{2}nR''(1)\right)^k}{k!}\delta_{s-2k,q}$
  - Accumulate code CIOWE: $A^{(acc)}_{q,\leq w} \leq \binom{n}{\lfloor q/2 \rfloor}\frac{w^{\lceil q/2 \rceil}}{\lceil q/2 \rceil !}$

$$\overline{A}^{(IRA)}_{p,\leq w} = \sum_{s,q} A^{(rep)}_{p,s} \frac{A^{(par)}_{s,q}}{\binom{nR'(1)}{s}} \frac{A^{(acc)}_{q,\leq w}}{\binom{n}{q}}$$

- Notice the $R''(1)$ in $A^{(par)}_{s,q}$
  - For this sequence, we find that $R''(1) = \Theta\left(M^{1/2}\right)$
  - For fixed n, we find $d_{min} \to 0$ as M increases
  - For fixed M, we find $d_{min} \geq n^{1/3-\varepsilon}$ as n increases
  - Fixed input wt., $n = \Omega\left(M^{3/2}\right)$ sufficient for $d_{min} \geq n^{1/3-\varepsilon}$

Conclusions

- Block length vs. gap to capacity for iterative decoding
    - The real obstacle for capacity achieving codes
- Finite length scaling law
    - Has great potential for this problem
        - Problem A: Parameters require numerical methods
        - Problem B: Non-uniform convergence
        - Can we get upper/lower bounds on n instead?
- Weight Enumerator Analysis
    - Required to prove convergence to zero erasures
        - Gives lower bounds on n
        - Needs refinement to prove $d_{min} = \Omega\left(n^{1/3-\varepsilon}\right)$