Compressed Sensing and Measurement Matrices with Large Symmetry Groups

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Acknowledgments

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  - Santhosh Kumar and Rob Calderbank
  - Santhosh Kumar, Shrinivas Kudekar, Marco Mondelli, Rüdiger Urbanke, and Eren Şaşoğlu
  - Galen Reeves
- Related papers [PR25, RP19, KKM+17, KCP16] can be found here:

  https://arxiv.org/abs/1601.04689

  https://arxiv.org/abs/1607.02524

  https://ieeexplore.ieee.org/abstract/document/7606832

  https://arxiv.org/abs/2504.15394

Outline

Compressed Sensing

- **Linear measurements of natural signals allow reconstruction from few samples.**
- Multiple landmark papers in 2006
  - Donoho ∼36K citations
  - Candes, Romberg, and Tao ∼20K
  - Too many to list with ∼9K citations
- Important connections to earlier work
  - Harmonic analysis and Kolmogorov widths
  - L1 regularization for geophysics in 1973
  - Basis pursuit and LASSO in the 1990s

Compressed Sensing

David L. Donoho, *Member, IEEE*

Robust Uncertainty Principles: Exact Signal Reconstruction From Highly Incomplete Frequency Information

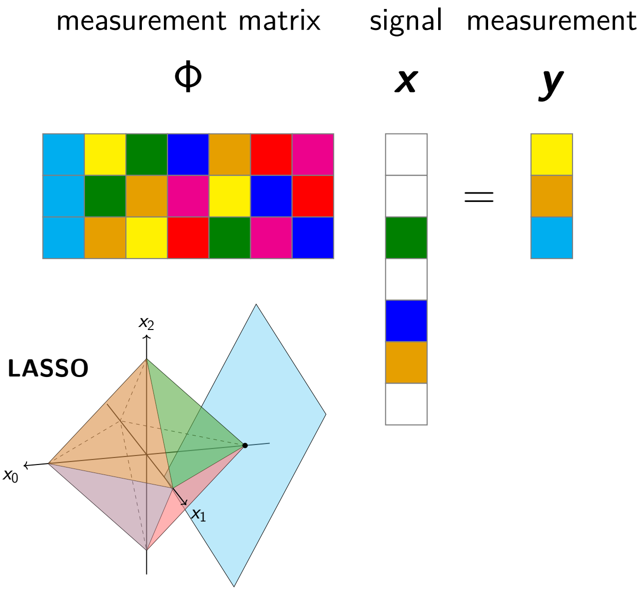
Emmanuel J. Candès, Justin Romberg, *Member, IEEE*, and Terence Tao

ROBUST MODELING WITH ERRATIC DATA†

JON F. CLAERBOUT* AND FRANCIS MUIR‡

Compressed Sensing: Figure 1

- measurement matrix: $\Phi \in \mathbb{R}^{M \times N}$
- signal: $\boldsymbol{x} \in \mathbb{R}^N$
- measurement: $\boldsymbol{y} = \Phi \boldsymbol{x}$
- decoding: $\hat{\boldsymbol{x}} = \underset{\boldsymbol{x} \in \mathbb{R}^n : \Phi \boldsymbol{x} = \boldsymbol{y}}{\arg\min} \|\boldsymbol{x}\|_1$
- matched to sparse signals
- e.g., with iid random Φ
- If noise, $\boldsymbol{y} = \Phi \boldsymbol{x} + \boldsymbol{w}$

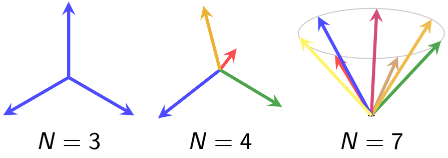


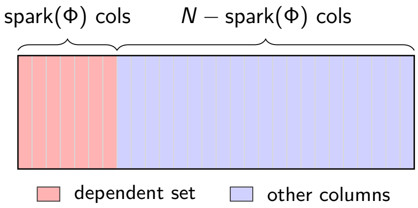
A Few Noteworthy Applications in Communications

- Capacity-achieving Sparse Superposition Codes via Approximate Message Passing Decoding [RGV17]
- SPARCs for Unsourced Random Access [FJC21]
- A Coded Compressed Sensing Scheme for Uncoordinated Multiple Access [ACN20]
- Joint Message Detection and Channel Estimation for Unsourced Random Access in Cell-Free User-Centric Wireless Networks [ÇGOC25]

Measurement Matrix Properties: Coherence and Spark

- **Setup:** $\Phi \in \mathbb{C}^{M \times N}$ has unit-norm columns.
- **Coherence:** max absolute inner product
$$\mu(\Phi) \;=\; \max_{j \neq k} \left|\langle \phi_j, \phi_k \rangle\right|.$$
- **Welch bound:** lower bound on coherence
$$\mu(\Phi) \;\geq\; \sqrt{\frac{M-N}{M}} \cdot \frac{1}{\sqrt{N-1}}.$$
equality iff equiangular tight frame (ETF)
- **Spark:** smallest # dependent columns
$$\text{spark}(\Phi) \;=\; \min\left\{|S| : \text{rank}(\Phi_S) < |S|\right\}.$$





Motivating Questions

- How can one design good Φ matrices?
  - **Random:** For iid Gaussian, small coherence and large spark
  - **Theory:** Vandermonde matrices and combinatorial designs
- What if entries restricted to small set (e.g., ± 1)?
  - Random matrices quite good but harder to analyze
  - Deterministic matrices hard to find and to analyze
- Can we replace randomness with something else?

Outline

What Can Symmetry Provide?

- Let Φ define a unit-norm tight frame: $\mathrm{diag}(\Phi^*\Phi) = I$ and $\Phi\Phi^* = \frac{N}{M} I$
- Φ has transitive symmetry group where stabilizer has no orbit size $< k$
- Then, the coherence satisfies

$$\mu(\Phi) \leq \sqrt{\frac{N - M}{k\,M}}$$

- Matches Welch bound with doubly transitive symmetry $k = N - 1$

Also, see earlier compressed sensing work on erasures and symmetry [FM12, JMF13, FJMP15, JK25].

Permutation Symmetry of Φ

- **Signed Permutation:** For $\pi \in \mathbb{S}_N$, matrix P_π gives

$$\Phi P_\pi = [\pm\phi_{\pi(1)}, \ldots, \pm\phi_{\pi(N)}]$$

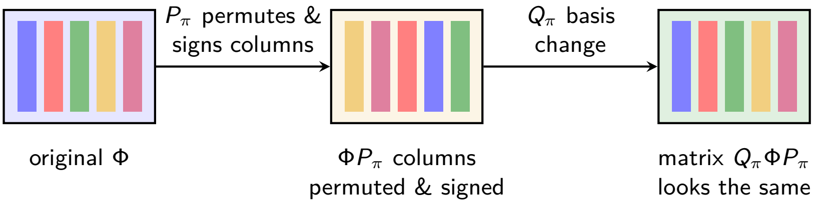
- **Symmetry group:**

$$\mathcal{G} = \{\pi \in \mathbb{S}_N \mid \exists Q_\pi,\ Q_\pi \Phi = \Phi P_\pi\}$$

- **Transitive:** $\mathcal{G}$ acts transitively on $\{1, \ldots, N\}$

Idea: For Φ, signed permutations P_π ($\pi \in \mathcal{G}$) of columns can be reversed by invertible change of basis on the left.

Idea: All columns equivalent.



Symmetric Unit-Norm Tight Frames (UNTFs)

Properties:

- **(A0) Unit-norm columns.** $\|\phi_j\|_2 = 1$ for all j. Hence the Gram matrix $G = \Phi^*\Phi$ satisfies $G_{jj} = 1$.
- **(A1) Orbit size.** Let $\mathcal{G}_0 = \{\pi \in \mathcal{G} : \pi(1) = 1\}$ act on $\{2, \ldots, N\}$. Every nontrivial $\mathcal{G}_0$-orbit has size $\geq k$.
- **(A2) Tight frame.** $\Phi\Phi^* = \frac{N}{M} I$ implies eigenvalues of G are N/M (mult. M) and 0 (mult. $N - M$). So,

$$\sum_{i \neq j} |G_{ij}|^2 = \|G\|_F^2 - \sum_j |G_{jj}|^2 = \frac{N^2}{M} - N = N \frac{N - M}{M}.$$

- **Orthogonal symmetry.** For a UNTF, all Q_π change of basis matrices must satisfy $Q_\pi^* Q_\pi = I$.

Stabilizer orbit lemma

Let $\mu \triangleq \max_{i \neq j} |G_{ij}|$. If $|G_{1j}| = \mu$ for some $j \neq 1$, then by (A1) there are $\geq k$ indices $j \neq 1$ such that

$$|G_{1j}| = \mu.$$

What we will prove next

$$\mu \geq \sqrt{\frac{N - M}{k\, M}}.$$

Coherence Lower Bound for Symmetric UNTF

Theorem

For a transitive UNTF with stabilizer orbits size $\geq k$,

$$\mu \leq \sqrt{\frac{N-M}{k\,M}}.$$

Proof.

Step 1. By transitive symmetry, all rows and columns of $G = \Phi^*\Phi$ are permutations of a fixed vector. Thus, $G_{ii} = 1$ implies:

$$\sum_{j\neq 1} |G_{1j}|^2 = \frac{1}{N}\sum_{i\neq j} |G_{ij}|^2.$$

Step 2. Eigenvalues of G give $\|G\|_F^2 = M\frac{N^2}{M^2}$ and

$$\frac{1}{N}\sum_{i\neq j} |G_{ij}|^2 = \frac{1}{N}\left(M\frac{N^2}{M^2} - N\right) = \frac{N-M}{M}.$$

Cont'd.

Step 3. The stabilizer orbit lemma shows $\geq k$ off-diagonal entries in row 1 have magnitude μ, so

$$\sum_{j\neq 1} |G_{1j}|^2 \;\geq\; k\,\mu^2.$$

Step 4. Combining gives $k\mu^2 \leq (N-M)/M$ and

$$\mu \leq \sqrt{\frac{N-M}{k\,M}}.$$

□

Comparison to Welch bound

Welch gives $\mu \geq \sqrt{\frac{N-M}{M(N-1)}}$. Our extension replaces $N-1$ by k and matches when doubly transitive. We assume tight frame, but Welch equality implies.

Outline

Connection to Erasure Channels

- In 2015, while analyzing a deterministic sequence of error-correcting codes, we realized that symmetry was the key.
- In a sense, code symmetry and random erasures replaces randomness in the code. This result required doubly transitive symmetry.
- With Santhosh Kumar and Rob Calderbank, this was extended to the stabilizer-orbit symmetry now used in this work.

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Reed–Muller Codes Achieve Capacity on Erasure Channels

Shrinivas Kudekar, Santhosh Kumar, *Student Member, IEEE*, Marco Mondelli, *Student Member, IEEE*, Henry D. Pfister, *Senior Member, IEEE*, Eren Şaşoğlu, and Rüdiger L. Urbanke, *Senior Member, IEEE*

2016 IEEE Information Theory Workshop (ITW)

Beyond Double Transitivity: Capacity-Achieving Cyclic Codes on Erasure Channels

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Reed–Solomon (RS) codes, Partial Fourier Matrices, and Paley ETFs

- Consider the $N = 7$ cyclic RS code over $\mathbb{F}_8$
  - let α be a primitive element of $\mathbb{F}_8$
  - parity-check matrix H is partial Fourier
  - recovers all patterns of at most 3 erasures

$$H = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 & \alpha^6 \\ 1 & \alpha^2 & \alpha^4 & \alpha^6 & \alpha & \alpha^3 & \alpha^5 \\ 1 & \alpha^3 & \alpha^6 & \alpha^2 & \alpha^5 & \alpha & \alpha^4 \end{bmatrix}$$

- Paley equiangular tight frame (ETF) $N = 7$
  - similar except $\phi = e^{2\pi i/7}$ and complex
  - prime DFT $\Rightarrow$ square submatrices invertible

$$\Phi = \begin{bmatrix} 1 & \phi & \phi^2 & \phi^3 & \phi^4 & \phi^5 & \phi^6 \\ 1 & \phi^2 & \phi^4 & \phi^6 & \phi & \phi^3 & \phi^5 \\ 1 & \phi^4 & \phi & \phi^5 & \phi^2 & \phi^6 & \phi^3 \end{bmatrix}$$

- What do erasure results say about ETFs?
  - As $M \to \infty$, symmetry $\Rightarrow$ almost all $M \times (1-\epsilon)M$ submatrices invertible?
  - Measurement design for erasures [FM12, JMF13, FJMP15, **?**]

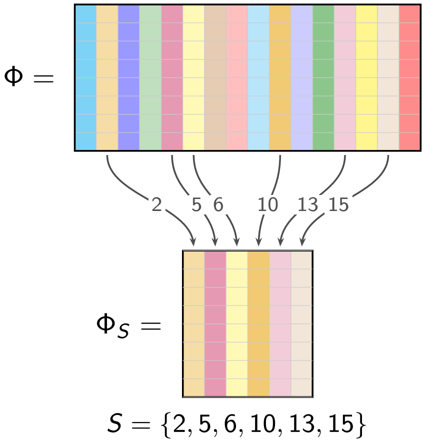
Signal Recovery from Erasures

- **Problem Setup:**
  - **Signal:** $\boldsymbol{x} \in \mathbb{R}^N$
  - **Matrix:** $\Phi \in \mathbb{R}^{M \times N}, \quad M = \delta N$
  - **Measurement:** $\boldsymbol{y} = \Phi \boldsymbol{x}$
- **Challenge:** subset $S \subseteq [N]$ of $\boldsymbol{x}$ erased
  - **Erasures:** We don't know subvector $\boldsymbol{x}_S$ but we do know all the rest $\boldsymbol{x}_{S^c}$

$$\boldsymbol{y} = \begin{bmatrix} \Phi_S & \Phi_{S^c} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_S \\ \boldsymbol{x}_{S^c} \end{bmatrix} = \Phi_S \, \boldsymbol{x}_S + \Phi_{S^c} \, \boldsymbol{x}_{S^c}$$

  - **Recovery:** If submatrix Φ_S is invertible, then we can compute

$$\boldsymbol{x}_S = \Phi_S^{-1} \left(\boldsymbol{y} - \Phi_{S^c} \, \boldsymbol{x}_{S^c} \right)$$



*First N naturals denoted by $[N] := \{1, 2, \ldots, N\}$.

Outline

Equiangular Tight Frames and the Kesten-MacKay Law

- Recovery with partial knowledge of support
  - Standard CS setup: $\boldsymbol{y} = \Phi \boldsymbol{x} + \boldsymbol{w}$
  - Given random set S containing true support
  - Noise amplification of linear reconstruction given by the eigenvalues of $P_S^\top \Phi^\top \Phi P_S$
- What is its eigenvalue distribution?
  - For Haar random Φ, it is MANOVA.
  - Conjecture: ETF $\Rightarrow$ MANOVA [HZG17]
  - Shown for real ETFs ($N = 2M$) [MMP21]
  - Established for general ETFs [K23]
- What does this imply? [DSL24]

Random subsets of structured deterministic frames have MANOVA spectra

Marina Haikin[a], Ram Zamir[a], and Matan Gavish[b,1]

Kesten–McKay Law for Random Subensembles of Paley Equiangular Tight Frames

Mark Magsino[1] · Dustin G. Mixon[1] · Hans Parshall[2]

Generic MANOVA limit theorems for products of projections

Dmitriy Kunisky*

UNIVERSALITY OF APPROXIMATE MESSAGE PASSING WITH SEMIRANDOM MATRICES

By Rishabh Dudeja[1,a], Yue M. Lu[2,c] and Subhabrata Sen[1,b]

Compressed Sensing with Orthogonal AMP and State Evolution

What is Orthogonal AMP (OAMP)?

- OAMP is a variation of AMP with a linear estimation (LE) step and a nonlinear denoising (NLD) step.
- Designed to improve performance for general Φ by enforcing decorrelation in the LE step and divergence free denoising in the NLD step.

What is state evolution (SE)?

- SE is a scalar recursion for the per-iteration MSE τ_t^2 before denoising and v_t^2 afterward.
- For i.i.d. Gaussian Φ, SE is a key result. For more general matrices, SE is still the subject of research.

Goal of this section

Justify OAMP SE using universality from Dudeja–Lu–Sen (Ann. Probab. 2023) when Φ is deterministic with symmetry.

Key mechanisms

Symmetric UNTF provides coherence decay while setup completes the [DLS23] semi-random matrix structure.

Mapping OAMP to Dudeja-Lu-Sen for UNTFs

OAMP algorithm for UNTF Φ:

- **LE step:** $\boldsymbol{r}_t = \boldsymbol{s}_t + W_t(\boldsymbol{y} - \Phi \boldsymbol{s}_t)$
- **NLD step:** $\boldsymbol{s}_{t+1} = \eta_t(\boldsymbol{r}_t)$ (elementwise)

$$\mathbb{E}[\eta_t'(X + \tau Z)] = 0 \text{ (divergence-free)}.$$

- MMSE LE for UNTF: $W_t = \Phi^\top$
- Errors $\boldsymbol{q}_t := \boldsymbol{s}_t - \boldsymbol{x}$ and $\boldsymbol{h}_t := \boldsymbol{r}_t - \boldsymbol{x}$ give

$$\boldsymbol{h}_t = (I - \Phi^\top \Phi)\boldsymbol{q}_t + \Phi^\top \boldsymbol{w}.$$

- For symmetric P_X, $\Phi = \Phi_0 S$ with diagonal random sign matrix S equal in distribution.

Memory-free iteration

[DLS23] analyzes the iteration

$$\boldsymbol{z}^{(t+1)} = M\left(f_{t+1}\left(\boldsymbol{z}^{(t)}\right)\right),$$

where $f_t : \mathbb{R} \to \mathbb{R}$ defined by NLD is applied elementwise and we $M = I - \Phi^\top \Phi$.

Semi-random ensemble [DLS23]

For all $\varepsilon > 0$,

$$\|M\|_\infty \lesssim N^{-1/2+\varepsilon}$$
$$\|M\|_{\text{op}} \lesssim 1$$
$$\max_{i \neq j} |(MM^\top)_{ij}| \lesssim N^{-1/2+\varepsilon}$$

State evolution for semi-random ensembles!

State Evolution for Incoherent Symmetric UNTFs

Definition

We say a UNTF sequence $\{\Phi\}$ is *incoherent* if, for all fixed $\varepsilon > 0$,

$$\max_{i \neq j} \left|(\Phi^\top \Phi)_{ij}\right| \lesssim N^{-1/2+\varepsilon} \qquad \text{as } N \to \infty.$$

Corollary ([DLS23])

Consider the OAMP recursion with

- *measurement* $\boldsymbol{y} = \Phi \boldsymbol{x} + \boldsymbol{w}$ *where* $\{\Phi\}$ *is a sequence of deterministic incoherent UNTFs;*
- *linear estimator given by LMMSE;*
- *scalar denoisers* η_t *that are continuously differentiable, Lipschitz, and divergence free.*

Then, for any fixed number of iterations, the empirical law of the OAMP error iterates converges to the law predicted by the standard OAMP state evolution recursion.

Symmetry implies state evolution because it gives incoherence for UNTFs.

Natural Open Questions

- Does the spectrum converge under weaker conditions on Φ?
  - For orbit size k, what rate of increase $k(N)$ suffices? If $k = N^{1-o(1)}$, then this follows from [Kun23]. What about $k = N^{\epsilon}$?
- Can one prove OAMP state evolution formula for slower $k(N)$ growth rate?
  - Already shown in [DLS23] for $k = N^{1-\varnothing(1)}$ but what about $k = \sqrt{N}$?
- Can one prove optimal recovery performance matches replica prediction?
  - Symmetry is likely sufficient for optimal recovery to match the statistical physics replica prediction but this is not known yet even for ETFs

Summary

- Lower bound on coherence via symmetry
- Implications for OAMP state evolution
- Connections with random matrix theory and open problems

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What Does Coherence Imply About Spark? [Tropp07]

- **Prop:** For any K-column submatrix X of Φ

$$\|X^\top X - I\|_2 \leq \ (K-1)\,\mu(\Phi).$$

Every K-subset independent if $(K-1)\mu < 1$

- **Coherence $\Rightarrow$ Spark:** lower bound

$$\mathrm{spark}(\Phi) \geq \ 1 + \frac{1}{\mu(\Phi)}.$$

- **Welch:** ETF spark lower bound $O(\sqrt{N})$.

- **Random Subsets:** ETF Φ_S well-conditioned

$$\text{if} \quad |S| = o\left(\frac{N}{\log N}\right)$$

ON THE CONDITIONING OF RANDOM SUBDICTIONARIES

JOEL A. TROPP