A Simple Proof of Threshold Saturation for Coupled Scalar Recursions

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Outline

Review of LDPC Codes

Spatially-Coupled LDPC Codes

Simple Proof of Threshold Saturation

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Spatially-Coupled LDPC Codes

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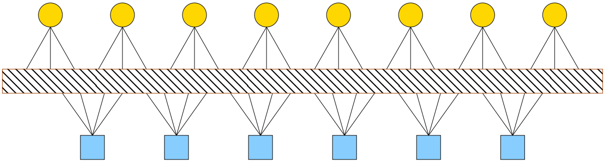
Low-Density Parity-Check (LDPC) Codes



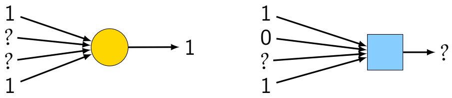
- Linear codes with a sparse parity-check matrix H
  - Regular (l, r): H has l ones per column and r ones per row
  - Irregular: number of ones given by degree distribution (λ, ρ)
  - Introduced by Gallager in 1960, but largely forgotten until 1995
- Tanner Graph
  - An edge connects check node i to bit node j if $H_{ij} = 1$
  - Naturally leads to message-passing iterative (MPI) decoding

Decoding LDPC Codes

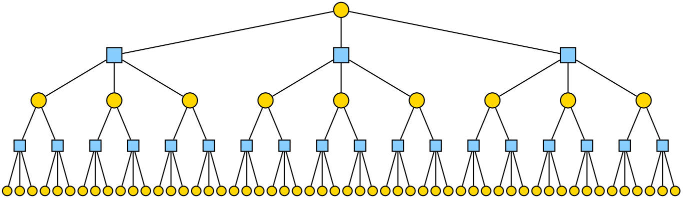
- Belief-Propagation (BP) Decoder
  - Low-complexity message-passing decoder introduced by Gallager
  - Local inference assuming all input messages are independent
- Density Evolution (DE)
  - Tracks distribution of messages during iterative decoding
  - BP noise threshold can be computed via DE
  - Long codes decode almost surely if DE predicts success
- Maximum A Posteriori (MAP) Decoder
  - Optimum decoder that chooses the most likely codeword
  - Infeasible in practice due to enormous number of codewords
  - MAP noise threshold can be bounded using EXIT curves

Message-Passing Iterative Decoding

- Constraint nodes define the valid patterns
  - Circles are bit nodes that assert all edges have same value
  - Squares are check nodes that assert sum of edge values is 0
- Iterative decoding on the binary erasure channel (BEC)
  - Msgs passed along edges in phases: bit-to-check and check-to-bit
  - Each output message depends only on the input messages
  - All messages are either correct value or erasure
- Message passing rules for the BEC
  - Bits pass the correct value unless all other inputs are erased
  - Checks pass the correct value only if all other inputs are correct

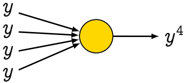


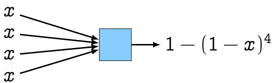
Computation Graph and Density Evolution



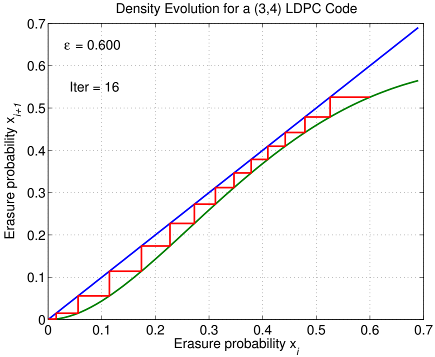
$$\hat{x}_3 = \varepsilon y_2^3$$
$$y_2 = 1-(1-x_2)^3$$
$$x_2 = \varepsilon y_1^2$$
$$y_1 = 1-(1-x_1)^3$$
$$x_1 = \varepsilon$$

- Computation graph for a (3,4)-regular LDPC code
  - Illustrates decoding from the perspective of a single bit-node
  - For long random LDPC codes, the graph is typically a tree
  - Allows density evolution to track message erasure probability
  - If x/y are erasure prob. of bit/check output messages, then





Density Evolution (DE) for LDPC Codes



Density evolution for a $(3,4)$-regular LDPC code:

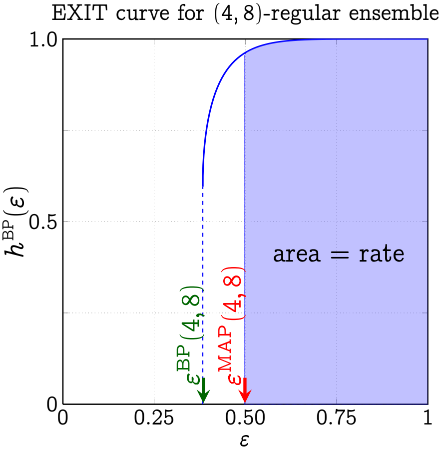
$$x_{i+1} = \varepsilon \left(1 - (1 - x_i)^3\right)^2$$

Decoding Threshold:

$$\varepsilon^* \approx 0.6474$$
$$\varepsilon^{\mathrm{Sh}} = 0.75$$

- Binary erasure channel (BEC) with erasure prob. ε
- DE tracks bit-to-check msg erasure rate x_i after i iterations
- Defines noise threshold $\varepsilon^{\mathrm{BP}}$ for the large system limit
    - Easily computed numerically for each code ensemble

EXtrinsic Information Transfer (EXIT) Curves



- Codeword $(X_1, \ldots, X_n)$
- Received $(Y_1, \ldots, Y_n)$

- Curve is extrinsic entropy $H(X_i | Y_{\sim i})$ vs. channel ε

- BP EXIT curve via DE
  - Ex. $h^{\text{BP}}(\varepsilon) = (x_\infty(\varepsilon))^4$
  - Equals 0 below BP thresh
  - Upper bounds MAP EXIT

- MAP EXIT
  - Equals 0 below MAP thresh
  - Area underneath equals rate

Spin, Inference, and Statistical Physics (1)

- Ising's Model of Magnetism
  - Magnetism caused by alignment of electron spins $\sigma_i \in \{+1, -1\}$
  - The system energy in an external field is modeled by

$$H(\sigma_1, \ldots, \sigma_n) = -\sum_{(i,j)\in\Lambda} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i$$

    for lattice Λ, spin coupling J_{ij}, and local field h_i

  - In equilibrium, the configuration probability is approximated by

$$P_\beta(\sigma_1, \ldots, \sigma_n) \propto e^{-\beta H(\sigma_1, \ldots, \sigma_n)}$$

- Binary Inference
  - Spin systems are mathematically similar to binary inference
  - Pairwise correlations in a binary vector controlled by J_{ij}
  - Observations encoded into the local magnetic fields h_i
  - The minimium-energy configuration is maximum a posteriori

Spin, Inference, and Statistical Physics (2)

- Phase Transitions
  - Inverse temperature $\beta = 1/T$ scales coupling and field strength
  - At high temperature ($\beta \to 0$), spin system resembles a liquid
  - At low temperature ($\beta \to \infty$), it can freeze into a ground state
  - The transition can be very complicated

- Statistical Physics of LDPC Codes
  - Code defined using generalized coupling coefficients J_α
  - Codewords are ordered crystalline structures
  - Field h_i is a function of Y_i and channel parameter
  - System is a supercooled liquid between BP and MAP threshold
    - Correct answer (crystalline state) has minimum energy w.h.p.
    - Spontaneous crystallization (i.e., decoding) does not occur w.h.p.

http://www.youtube.com/watch?v=Xe8vJrIvDQM

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Review of LDPC Codes

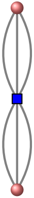
Spatially-Coupled LDPC Codes

Simple Proof of Threshold Saturation

Spatially-Coupled Codes: Background

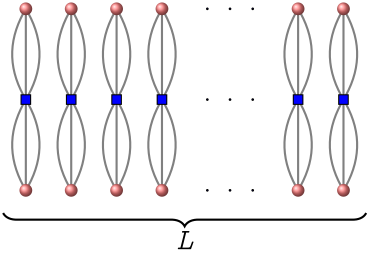
- LDPC Convolutional Codes were introduced by Felstrom and Zigangirov in 1999
- In 2005, LSZC showed that terminated regular LDPC convolutional codes have BP thresholds close to capacity
- Recently, KRU observed a general phenomenon whereby the BP threshold of spatially-coupled (SC) LDPC codes saturates to the "MAP" threshold of their uncoupled cousins
- This observation implies spatial coupling might benefit applications where iterative decoding falls short of MAP decoding

Spatial Coupling: The (l, r, L) Protograph Ensemble



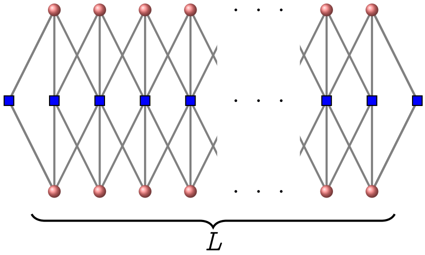
Protograph for $(3, 6)$-regular ensemble

Spatial Coupling: The (l, r, L) Protograph Ensemble



Chain of L protographs
for a $(3, 6)$-regular ensemble

Spatial Coupling: The (l, r, L) Protograph Ensemble



$(3, 6, L)$ SC protograph for
a coupled chain of $(3, 6)$ ensembles

$$\mathbf{H} = \begin{pmatrix} 110000000000\ldots000000 \\ 111100000000\ldots000000 \\ 111111000000\ldots000000 \\ 001111110000\ldots000000 \\ 000011111100\ldots000000 \\ 000000111111\ldots000000 \\ 000000001111\ldots110000 \\ 000000000011\ldots111100 \\ 000000000000\ldots111111 \end{pmatrix} \updownarrow L$$

Protograph parity-check matrix
before lifting

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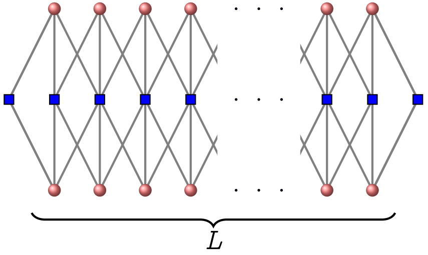

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Lift the protograph
Each node/edge copied M times

Each 1 becomes an
$M \times M$ permutation matrix

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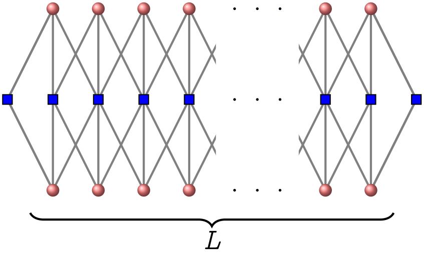

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Each 1 becomes an
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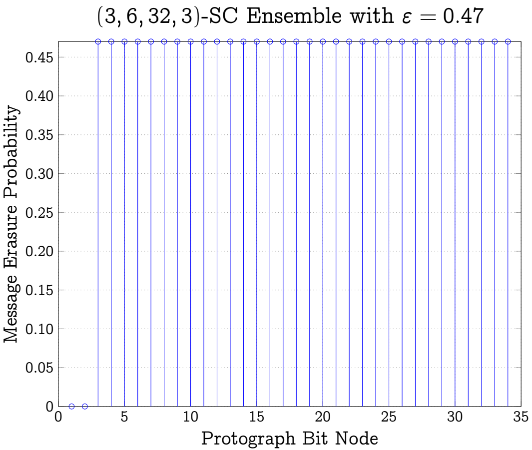
- (l, r, L) protograph ensemble has very regular structure
- (l, r, L, w)-SC ensemble randomizes edges over window size w

Density Evolution for the (l, r, L, w)-SC Ensemble



$$z_i^{(\ell+1)} = \varepsilon \left(1 - \frac{1}{w} \sum_{j=0}^{w-1} \left(1 - \frac{1}{w} \sum_{k=0}^{w-1} z_{i+j-k}^{(\ell)} \right)^{r-1} \right)^{l-1}$$

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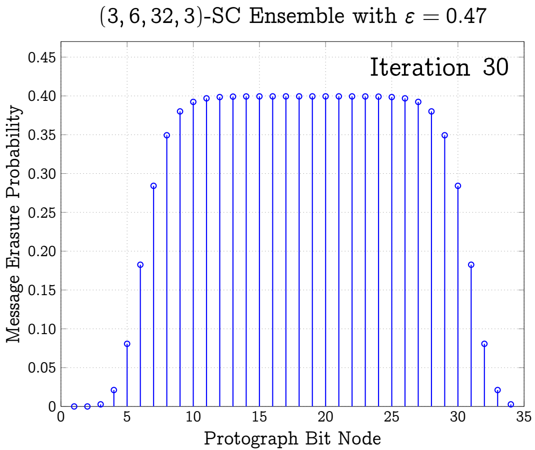

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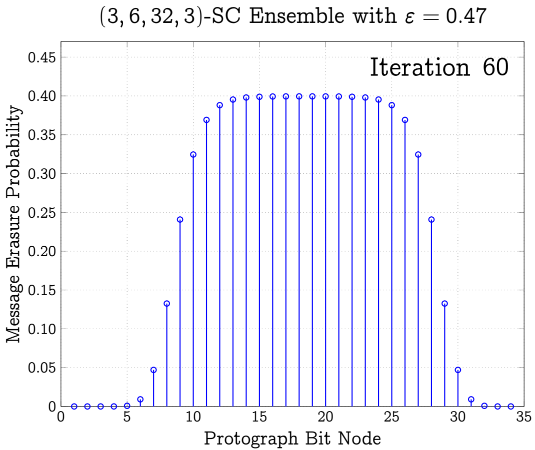

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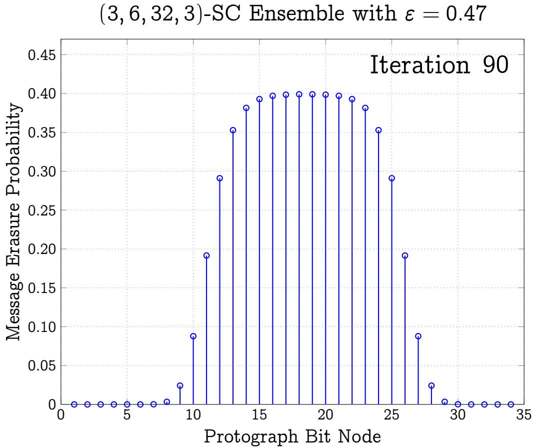

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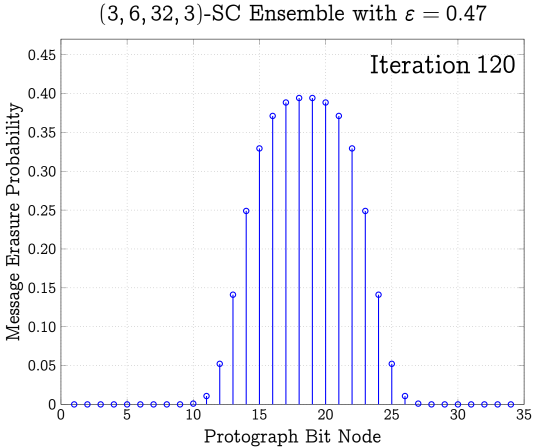

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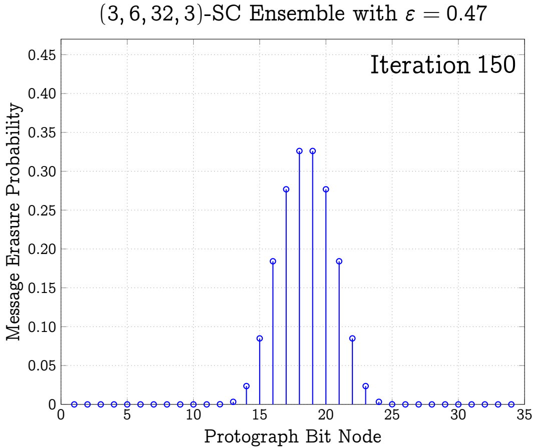

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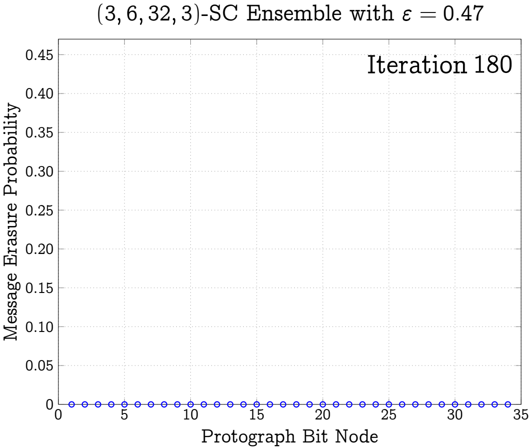

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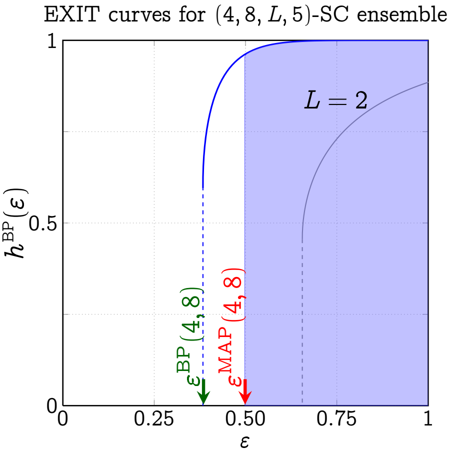
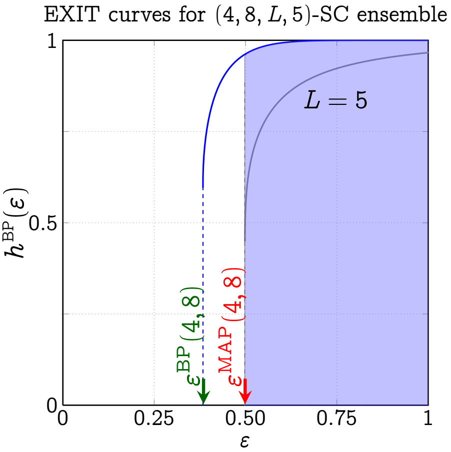
Threshold Saturation



- For finite L, SC ensemble has lower rate and higher threshold
  - As $L \to \infty$, code rate increases to original rate
  - BP Threshold decreases to MAP threshold of original

Threshold Saturation



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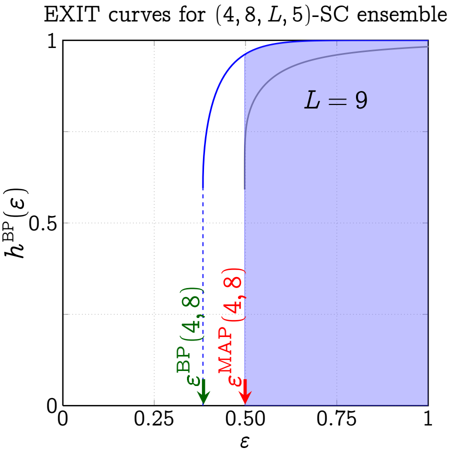

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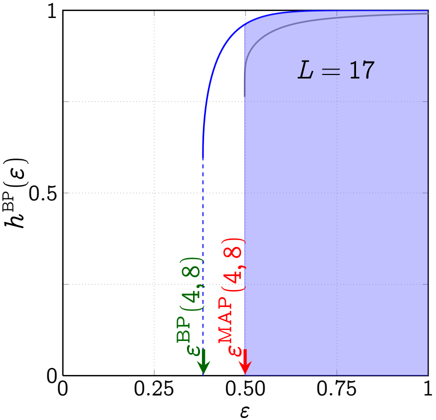
Threshold Saturation

EXIT curves for $(4, 8, L, 5)$-SC ensemble



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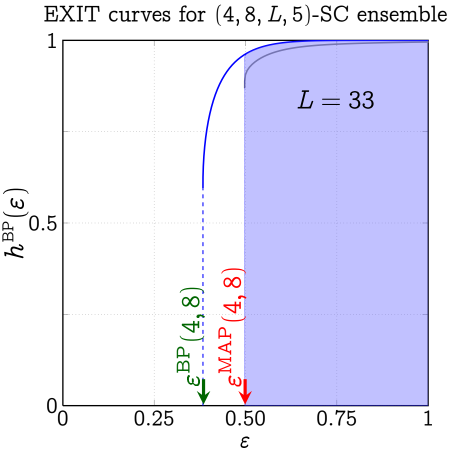

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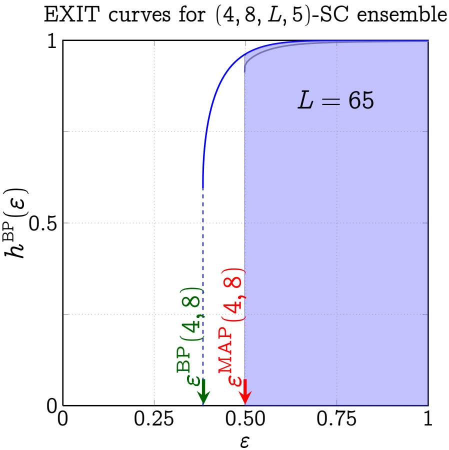

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Spatially-Coupled LDPC Codes

Simple Proof of Threshold Saturation

A Simple Proof of Threshold Saturation: Outline for Scalar Recursions

- Statement: The threshold of coupled scalar recursions increases to an intrinsic constant defined by the scalar recursion
- Proof Outline:
  1. Define the natural potential function for a scalar recursion
     - Show BP and MAP thresholds can be computed from potential
  2. Derive the potential function for the coupled scalar recursion
  3. Upper bound original system by modified recursion
  4. Show that, below MAP threshold, only fixed point is zero vector

Chronicle of Threshold Saturation Proofs

- For the BEC by KRU in 2010
  - Established many properties and tools used by later approaches
- For CDMA systems with GA by TTK in 2011
  - Our use of potential functions was motivated by this paper
- For compressed sensing with GA by DJM in 2011
  - Using a vector potential function in the continuous limit
- For general scalar recursions by Richardson in 2011
  - Informally reported a proof based on the continuous limit
- For BMS channels by KRU in 2012
  - Can our simplified approach work here?

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Admissible Scalar Recursion

$$x^{(\ell+1)} = f(g(x^{(\ell)}); \varepsilon)$$

- $f : [0,1] \times [0,1] \rightarrow [0,1]$
- $g : [0,1] \rightarrow [0,1]$
- $x^{(0)} = 1$ implies $x^{(\ell)} \in [0,1]$ for $\ell = 1, 2, \ldots$
- Recursion has parameter ε
  - Single recursion converges to 0 for ε below BP threshold
- f strictly increasing and $g'(x) > 0$ for $x, \varepsilon \in (0,1)$
- Both f, g twice differentiable with bounded derivatives
- Boundary conditions: $f(0;\varepsilon) = f(x;0) = g(0) = 0$

Density Evolution for the (3,6) LDPC Ensemble

$$x^{(\ell+1)} = \varepsilon(1-(1-x^{(\ell)})^5)^2$$

- $f(x;\varepsilon) = \varepsilon x^2$ (strictly increasing in ε, x)
- $g(x) = 1-(1-x)^5$ ($g'(x) = 5(1-x)^4 > 0$ for $x \in (0,1)$)
- Satisfies $f(0;\varepsilon) = f(x;0) = g(0) = 0$

- For $\varepsilon < 0.4294$, no fixed point for $x \in (0,1]$ implies $x^{(\ell)} \to 0$
- For $\varepsilon > 0.4295$, fixed point appears and $x^{(\ell)} \to x^{(\infty)} > 0.2652$

Single-System Potential and Thresholds

- Let the potential function $U(x;\varepsilon)$ of the recursion be

$$U(x;\varepsilon) \triangleq \int_0^x \Big(z - f(g(z);\varepsilon)\Big) g'(z) \mathrm{d}z$$
$$= xg(x) - G(x) - F(g(x);\varepsilon), \qquad (1)$$

where $F(x;\varepsilon) = \int_0^x f(z;\varepsilon)\mathrm{d}z$ and $G(x) = \int_0^x g(z)\mathrm{d}z$.

- Let the single-system threshold be

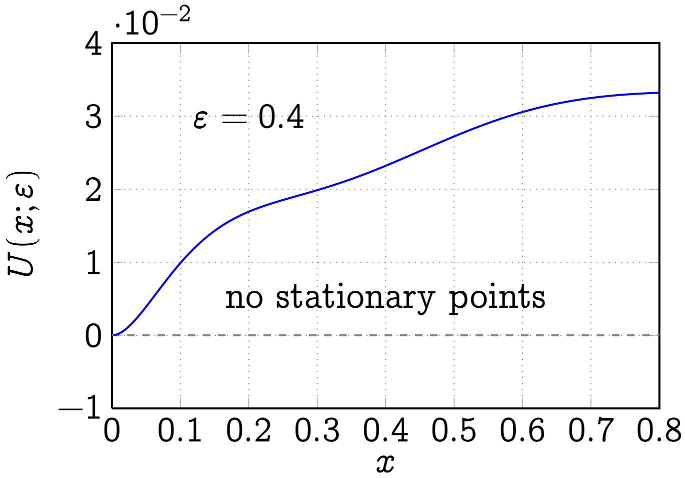
$$\varepsilon_s^* = \sup\{\varepsilon \in [0,1] \mid U'(x;\varepsilon) \geq 0 \; \forall \, x \in [0,1]\}$$

- Let the potential threshold be

$$\varepsilon^* = \sup\{\varepsilon \in [0,1] \mid U(x;\varepsilon) \geq 0 \; \forall \, x \in [0,1]\}$$

Potential Function for the (3,6) LDPC Ensemble

$$U(x;\varepsilon) = x(1-(1-x)^5) - \frac{6x-1+(1-x)^6}{6} - \varepsilon\frac{(1-(1-x)^5)^3}{3}$$



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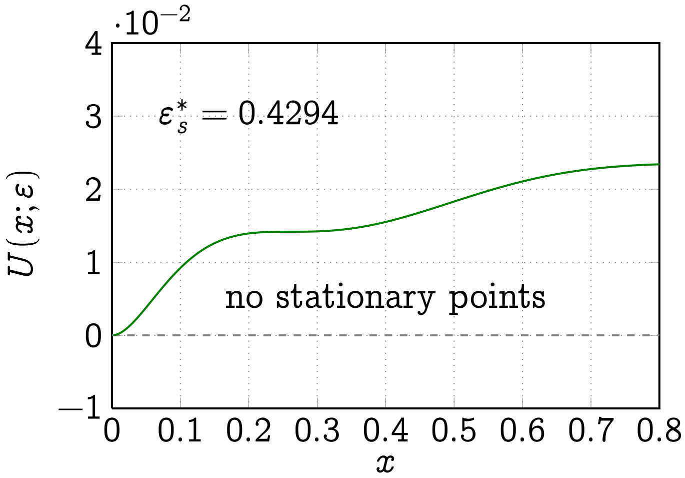

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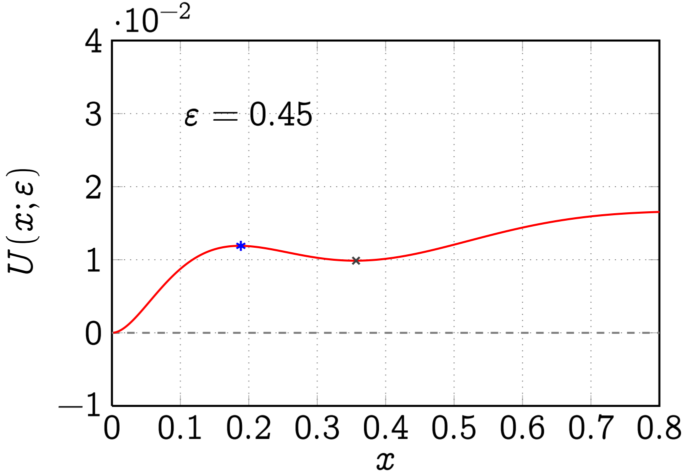

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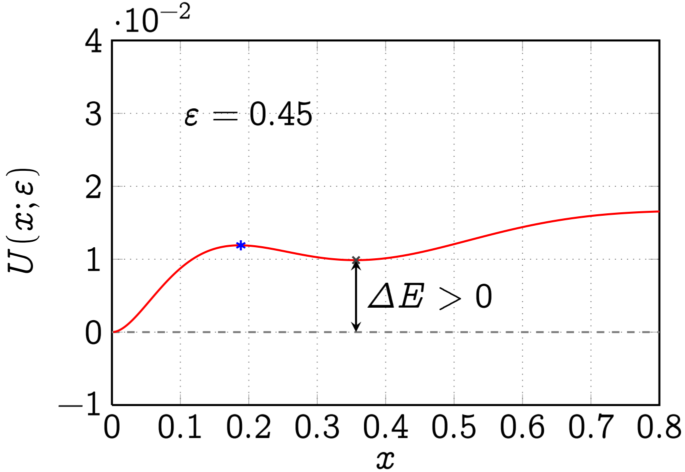

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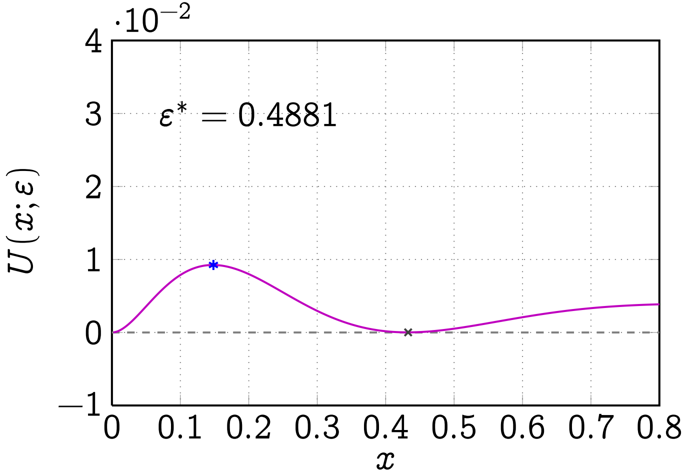

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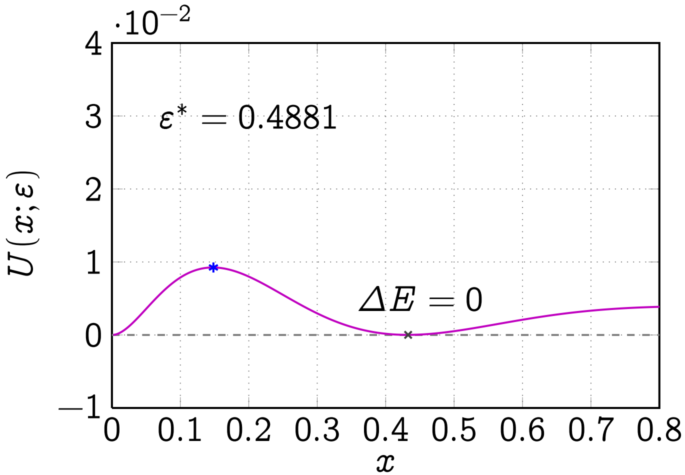

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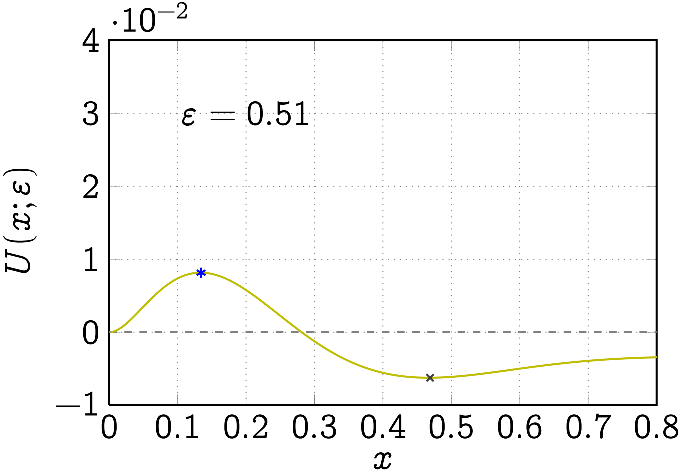

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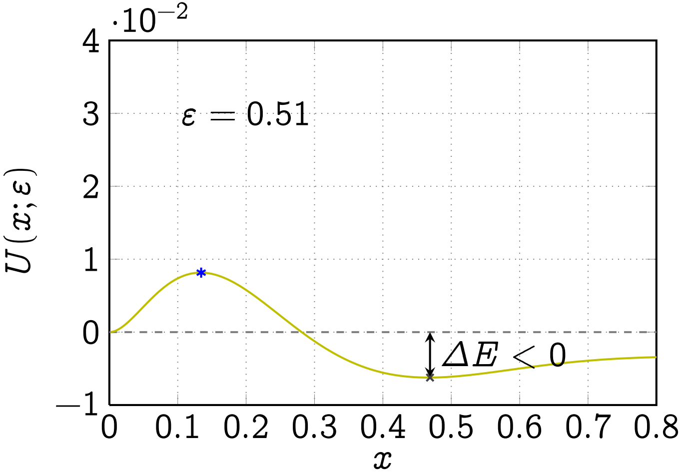
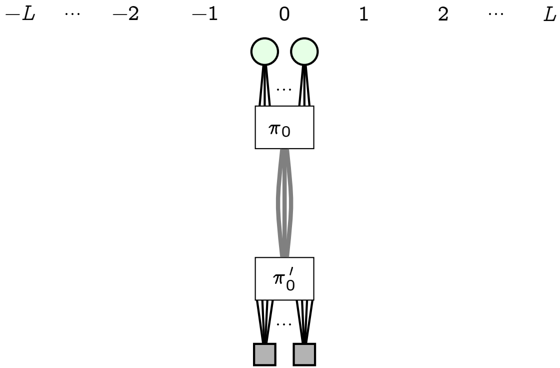


(l, r, L, w) Spatially-Coupled Ensemble



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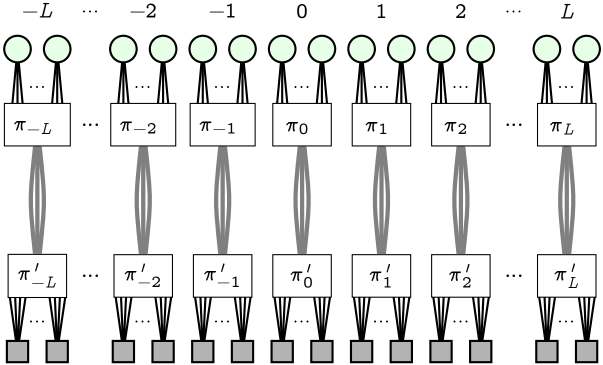

- Shown for $l = 3$, $r = 4$, and $w = 3$
- $2L + 1$ bit-node groups at positions $\mathcal{L}_f \triangleq \{-L, -L + 1, \ldots, L\}$
- $2L + w$ check-node groups at positions $\mathcal{L}_g \triangleq \{-L, \ldots, L + w - 1\}$

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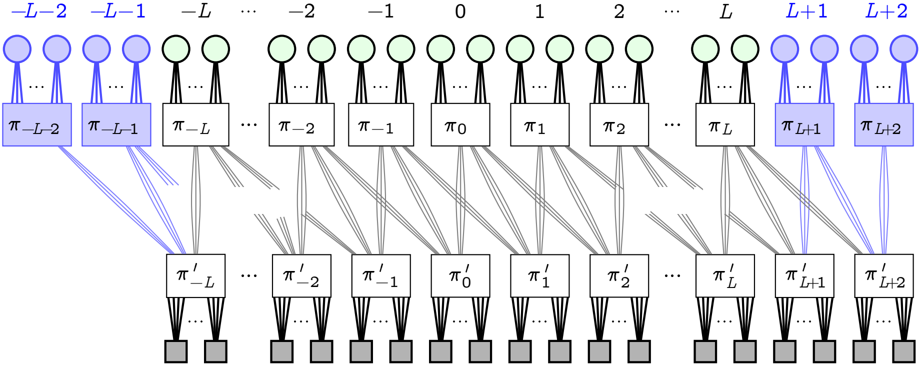

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Coupled Recursions

$$x_i^{(\ell+1)} = \frac{1}{w} \sum_{k=0}^{w-1} f\left(\frac{1}{w} \sum_{j=0}^{w-1} g\left(x_{i+j-k}^{(\ell)}\right); \varepsilon_{i-k} \right)$$

$$x_i^{(0)} = 1, i \in \{-L, \cdots, L+w-1\}$$

$$x_i^{(\ell)} = 0, i \notin \{-L, \cdots, L+w-1\}$$

$$\varepsilon_i = \varepsilon \mathbb{1}_{\{-L,\cdots,L\}}(i)$$

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$$\left.\begin{aligned} x_i^{(0)} &= 1, i \in \{-L, \cdots, \lfloor \tfrac{w-1}{2} \rfloor\} \\ x_i^{(\ell)} &= 0, i < -L \\ x_i^{(\ell)} &= x_0^{(\ell)}, i > \lfloor \tfrac{w-1}{2} \rfloor \end{aligned}\right\} \text{modified recursion}$$

Modified Coupled Recursion

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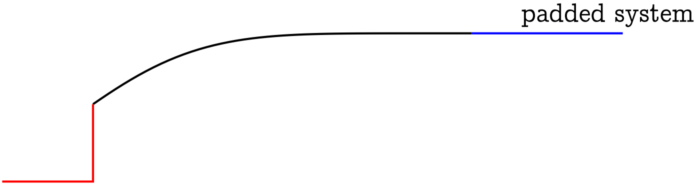

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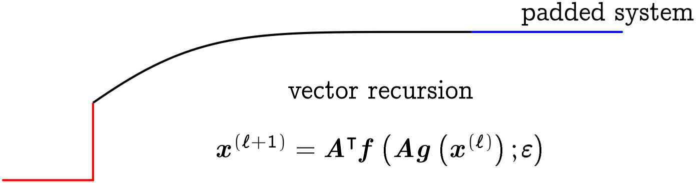

$$\boldsymbol{x}^{(\ell+1)} = \boldsymbol{A}^{\mathsf{T}} \boldsymbol{f}\left(\boldsymbol{A}\boldsymbol{g}\left(\boldsymbol{x}^{(\ell)}\right);\varepsilon\right)$$

$$\boldsymbol{A} = \frac{1}{w}\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

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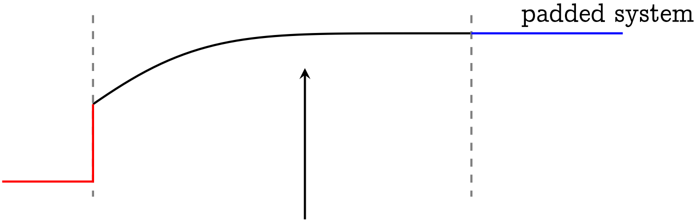


vector recursion gives exact one-step update for $i \in \mathcal{L}$

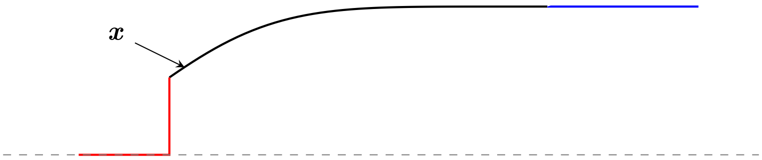
Potential for Vector Recursions

$$U(\boldsymbol{x};\varepsilon) = \int_C \boldsymbol{g}'(\boldsymbol{z})(\boldsymbol{z} - \boldsymbol{A}^\intercal \boldsymbol{f}(\boldsymbol{A}\boldsymbol{g}(\boldsymbol{z}))) \cdot \mathrm{d}\boldsymbol{z}$$
$$= \boldsymbol{g}(\boldsymbol{x})^\intercal \boldsymbol{x} - G(\boldsymbol{x}) - F(\boldsymbol{A}\boldsymbol{g}(\boldsymbol{x});\varepsilon)$$

- $G(\boldsymbol{x}) = \sum_i G(x_i)$, $F(\boldsymbol{x}) = \sum_i F(x_i)$
- $U'(\boldsymbol{x};\varepsilon) = \boldsymbol{g}'(\boldsymbol{z})(\boldsymbol{z} - \boldsymbol{A}^\intercal \boldsymbol{f}(\boldsymbol{A}\boldsymbol{g}(\boldsymbol{z})))$
- For coupled potential, Hessian bounded $\|U''(\boldsymbol{x};\varepsilon)\|_\infty \leq K_{f,g}$

Properties of Shift Matrix

Shift matrix $\boldsymbol{S}$ = right-shift in picture



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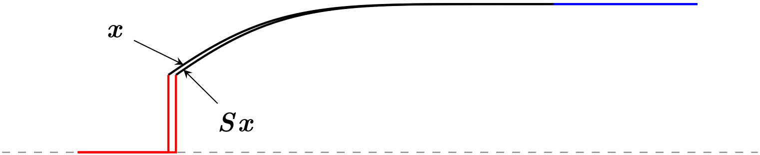

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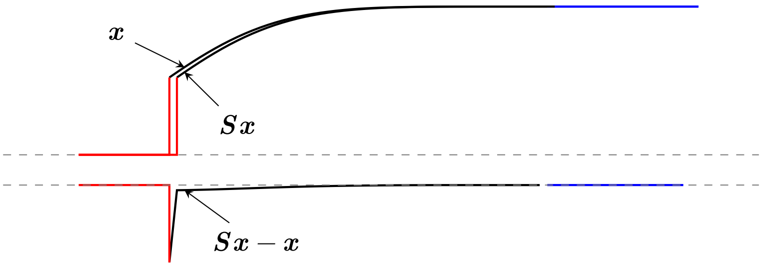

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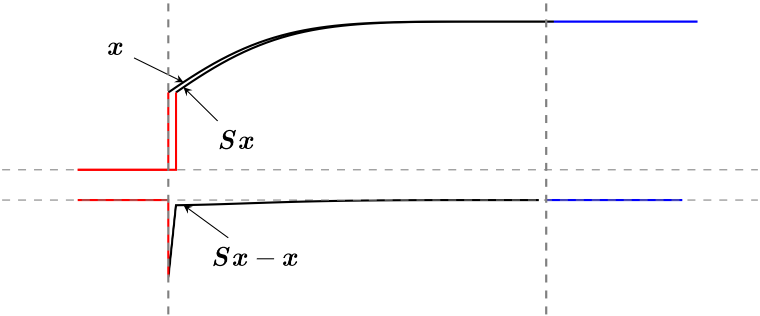

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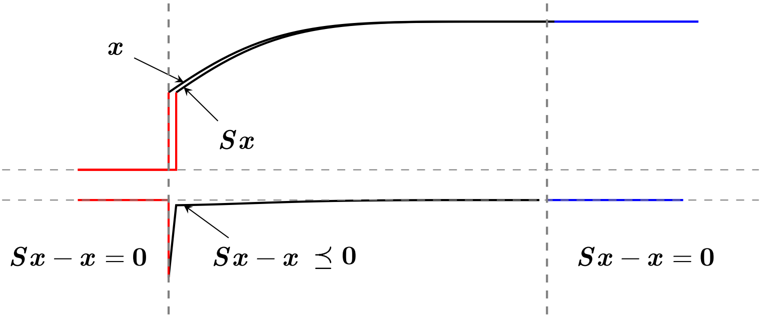

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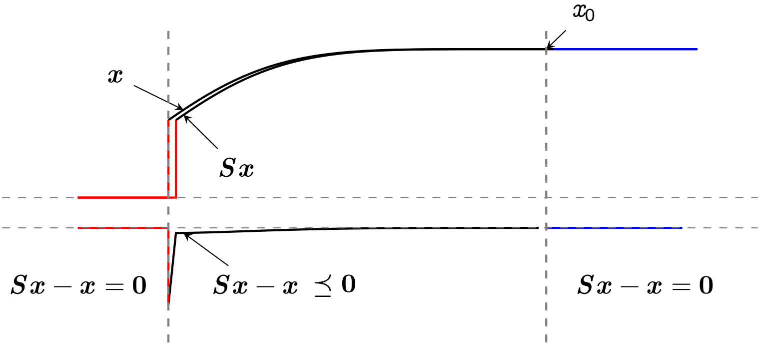

$$\|Sx - x\|_\infty \leq \frac{1}{w} \qquad \|Sx - x\|_1 = x_0$$

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$$
\begin{aligned}
U'(\boldsymbol{x};\varepsilon)\cdot(\boldsymbol{S}\boldsymbol{x}-\boldsymbol{x}) &= U(\boldsymbol{S}\boldsymbol{x};\varepsilon) - U(\boldsymbol{x};\varepsilon) \\
&\quad - \int_0^1 (1-t)(\boldsymbol{S}\boldsymbol{x}-\boldsymbol{x})^{\intercal} U''(\boldsymbol{x}(t);\varepsilon)(\boldsymbol{S}\boldsymbol{x}-\boldsymbol{x})\mathrm{d}t \\
&\leq \Delta U + \left| \int_0^1 (1-t)(\boldsymbol{S}\boldsymbol{x}-\boldsymbol{x})^{\intercal} U''(\boldsymbol{x}(t);\varepsilon)(\boldsymbol{S}\boldsymbol{x}-\boldsymbol{x})\mathrm{d}t \right| \\
&\leq \Delta U + \frac{1}{2}\|\boldsymbol{S}\boldsymbol{x}-\boldsymbol{x}\|_1 \max_{t\in[0,1]} \|U''(\boldsymbol{x}(t);\varepsilon)\|_\infty \|\boldsymbol{S}\boldsymbol{x}-\boldsymbol{x}\|_\infty \\
&\leq -U(x_{i_0};\varepsilon) + \frac{1}{2}\frac{x_{i_0}}{w} \max_{t\in[0,1]} \|U''(\boldsymbol{x}(t);\varepsilon)\|_\infty \\
&\leq -U(x_{i_0};\varepsilon) + \frac{K_{f,g}}{2w} < -U(x_{i_0};\varepsilon) + \Delta E(\varepsilon) \leq 0,
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- So, $\boldsymbol{x} = \boldsymbol{0}$ is the only f.p. of the modified coupled recursion
- The same conclusion holds for the original recursion because it is upper bounded by the modified recursion

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  - Simple proof of threshold saturation for scalar recursions
- For many multiuser problems,
  - Optimized LDPC codes are not universal
  - But suboptimal decoding is the main problem
  - Spatially-coupled joint decoders appear to be universal
    - Observed in general and now proven for some erasure models

Thanks for your attention