From BP to MAP via Spatial Coupling

Henry D. Pfister

joint work with Arvind Yedla, Phong Nguyen,
Yung-Yih Jian, and Krishna Narayanan

University of Texas, Austin

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Outline

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Review of LDPC Codes

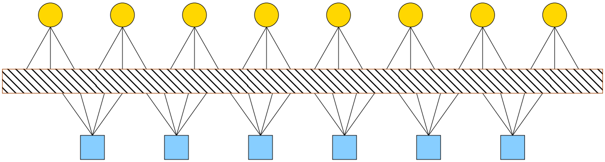
Spatially-Coupled LDPC Codes

Universality for Multiuser Scenarios

A Noisy Slepian-Wolf Problem

Simple Proof of Threshold Saturation

Low-Density Parity-Check (LDPC) Codes

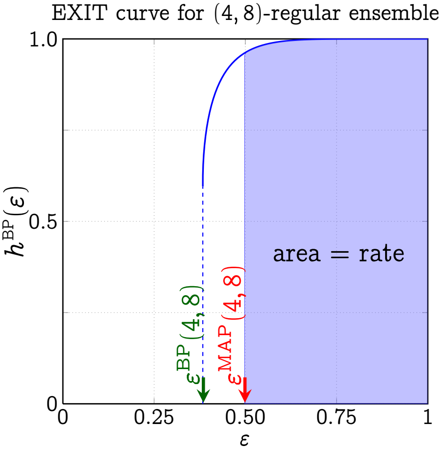


- Linear codes with a sparse parity-check matrix H
  - Regular (l, r): H has l ones per column and r ones per row
  - Irregular: number of ones given by degree distribution (λ, ρ)
  - Introduced by Gallager in 1960, but largely forgotten until 1995
- Tanner Graph
  - An edge connects check node i to bit node j if $H_{ij} = 1$
  - Naturally leads to message-passing iterative (MPI) decoding

Decoding LDPC Codes

- Belief-Propagation (BP) Decoder
  - Low-complexity message-passing decoder introduced by Gallager
  - Local inference assuming all input messages are independent
- Density Evolution (DE)
  - Tracks distribution of messages during iterative decoding
  - BP noise threshold can be computed via DE
  - Long codes decode almost surely if DE predicts success
- Maximum A Posteriori (MAP) Decoder
  - Optimum decoder that chooses the most likely codeword
  - Infeasible in practice due to enormous number of codewords
  - MAP noise threshold can be bounded using EXIT curves

EXtrinsic Information Transfer (EXIT) Curves



- Codeword $(X_1, \ldots, X_n)$
- Received $(Y_1, \ldots, Y_n)$
- Curve is extrinsic entropy $H(X_i | Y_{\sim i})$ vs. channel ε
- BP EXIT curve via DE
  - Ex. $h^{\mathrm{BP}}(\varepsilon) = (x_\infty(\varepsilon))^4$
  - Equals 0 below BP thresh
  - Upper bounds MAP EXIT
- MAP EXIT
  - Equals 0 below MAP thresh
  - Area underneath equals rate

Spin, Inference, and Statistical Physics (1)

- The Ising Model of Magnetism
  - Magnetism caused by alignment of electron spins $\sigma_i \in \{+1, -1\}$
  - The system energy in an external field is modeled by

$$H(\sigma_1, \ldots, \sigma_n) = -\sum_{(i,j)\in\Lambda} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i$$

  for lattice Λ, spin coupling J_{ij}, and local field h_i

  - In equilibrium, the configuration probability is approximated by

$$P_\beta(\sigma_1, \ldots, \sigma_n) \propto e^{-\beta H(\sigma_1, \ldots, \sigma_n)}$$

- Binary Inference
  - Spin systems are mathematically similar to binary inference
  - Pairwise correlations in a binary vector controlled by J_{ij}
  - Observations encoded into the local magnetic fields h_i
  - The minimium-energy configuration is maximum a posteriori

Spin, Inference, and Statistical Physics (2)

- Phase Transitions
  - Inverse temperature $\beta = 1/T$ scales coupling and field strength
  - At high temperature ($\beta \to 0$), spin system resembles a liquid
  - At low temperature ($\beta \to \infty$), it can freeze into a ground state
  - The transition can be very complicated

- Statistical Physics of LDPC Codes
  - Code defined using generalized coupling coefficients J_α
  - Codewords are ordered crystalline structures
  - Field h_i is a function of Y_i and channel parameter
  - System is a supercooled liquid between BP and MAP threshold
    - Correct answer (crystalline state) has minimum energy w.h.p.
    - Spontaneous crystallization (i.e., decoding) does not occur w.h.p.

http://www.youtube.com/watch?v=Xe8vJrIvDQM

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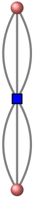
Spatially-Coupled Codes: Background

- LDPC Convolutional Codes were introduced by Felstrom and Zigangirov in 1999
- In 2005, LSZC showed that terminated regular LDPC convolutional codes have BP thresholds close to capacity
- Recently, KRU observed a general phenomenon whereby the BP threshold of spatially-coupled (SC) LDPC codes saturates to the "MAP" threshold of their uncoupled cousins
- This observation implies spatial coupling might benefit applications where iterative decoding falls short of MAP decoding

Spatial Coupling: The (l, r, L) Protograph Ensemble



Protograph for $(3, 6)$-regular ensemble

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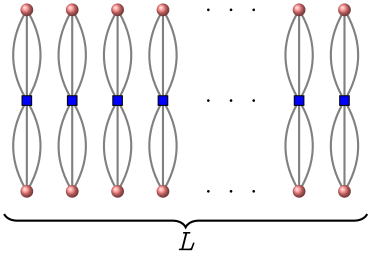
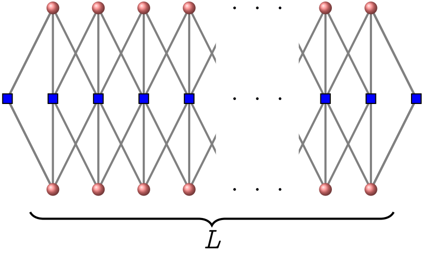


Chain of L protographs
for a $(3,6)$-regular ensemble

Spatial Coupling: The (l, r, L) Protograph Ensemble



$(3, 6, L)$ SC protograph for a coupled chain of $(3, 6)$ ensembles

$$\mathbf{H} = \begin{pmatrix} 110000000000\ldots000000 \\ 111100000000\ldots000000 \\ 111111000000\ldots000000 \\ 001111110000\ldots000000 \\ 000011111100\ldots000000 \\ 000000111111\ldots000000 \\ 000000001111\ldots110000 \\ 000000000011\ldots111100 \\ 000000000000\ldots111111 \end{pmatrix} \updownarrow L$$

Protograph parity-check matrix before lifting

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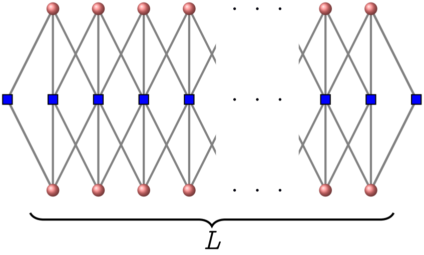

Lift the protograph
Each node/edge copied M times

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$M \times M$ permutation matrix

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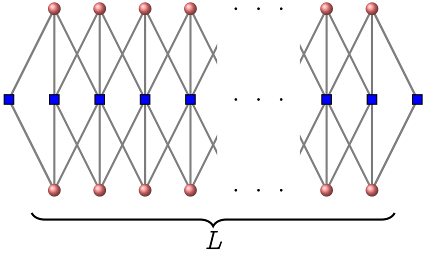

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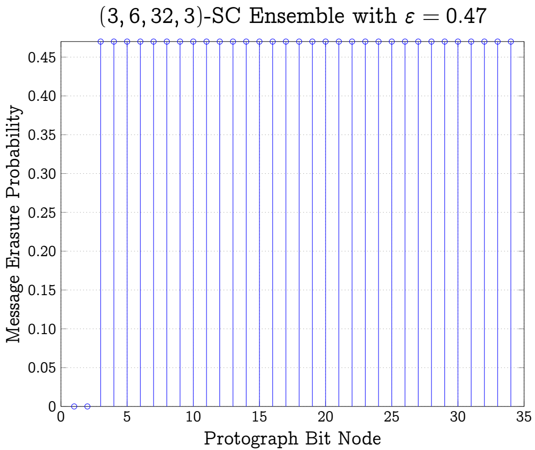
- (l, r, L) protograph ensemble has very regular structure
- (l, r, L, w)-SC ensemble randomizes edges over window size w

Density Evolution for the (l, r, L, w)-SC Ensemble



$$z_i^{(\ell+1)} = \varepsilon \left(1 - \frac{1}{w} \sum_{j=0}^{w-1} \left(1 - \frac{1}{w} \sum_{k=0}^{w-1} z_{i+j-k}^{(\ell)} \right)^{r-1} \right)^{l-1}$$

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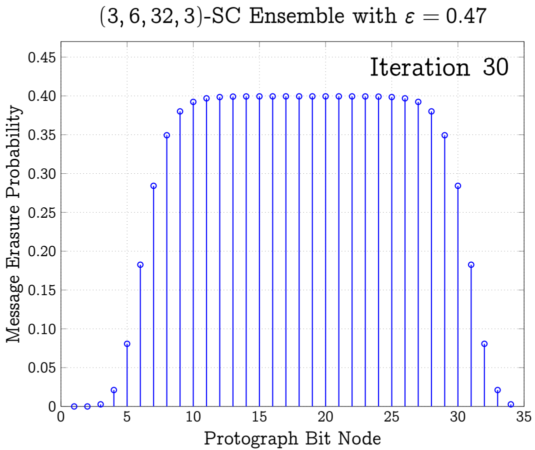

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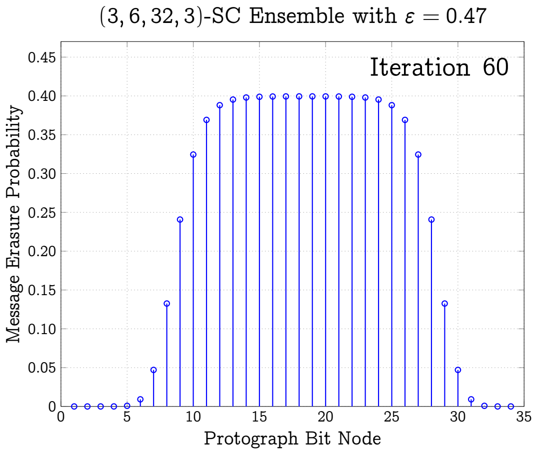

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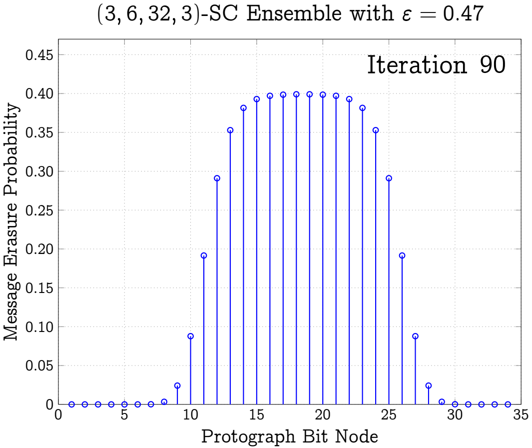

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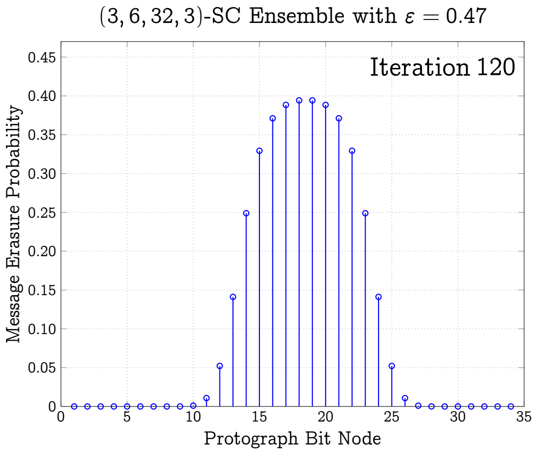

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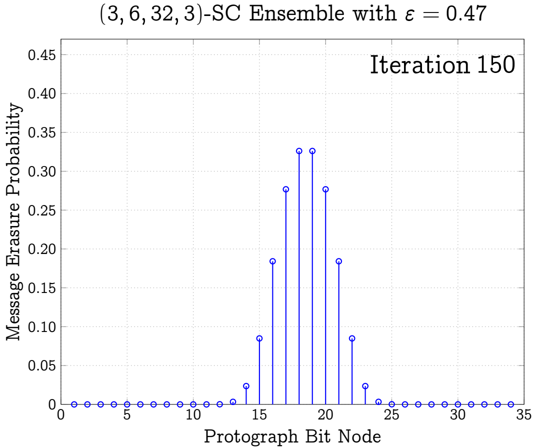

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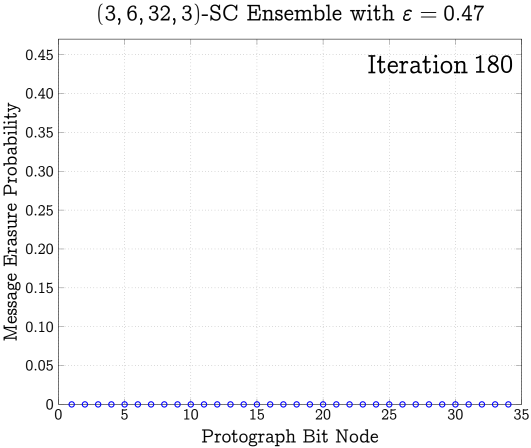

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Application: Universality

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Its achievable channel parameter region (ACPR) is the largest ACPR possible (i.e., the entire “capacity” region)

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  - ordered BMS family: optimized LDPC codes are nearly universal
  - all BMS channels: spatial-coupling provides provable universality

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- Multiuser communication
  - Multiple channels: ordered BMS families provide a partial order
  - For many problems (e.g., noisy SW and MAC)
    - Random codes with ML decoding are universal
  - Slepian-Wolf has multiple effective channels
  - MAC problem has multiple user gains

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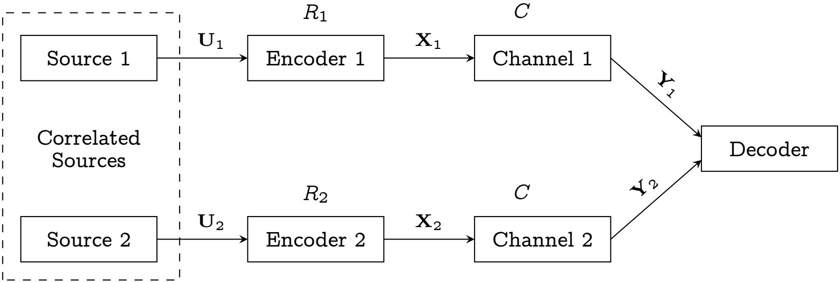
A Noisy Slepian-Wolf Problem



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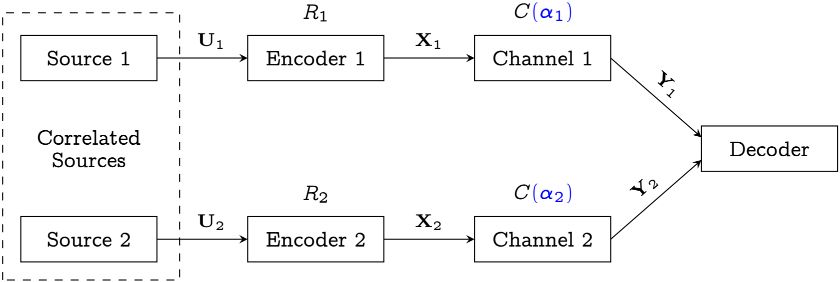

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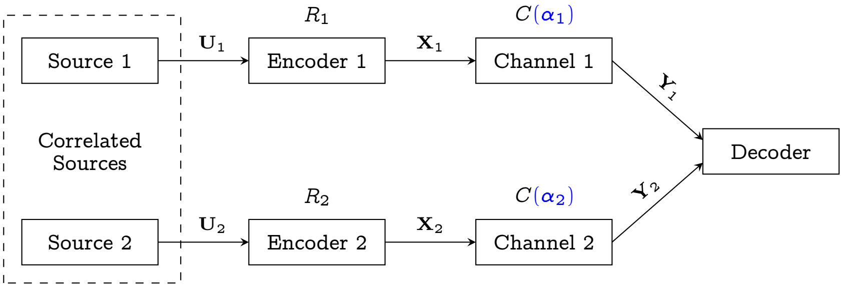

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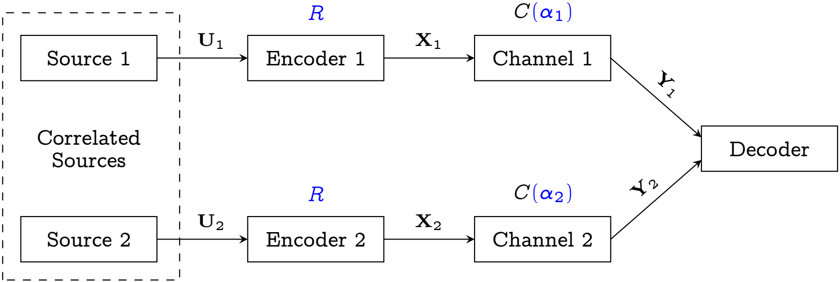

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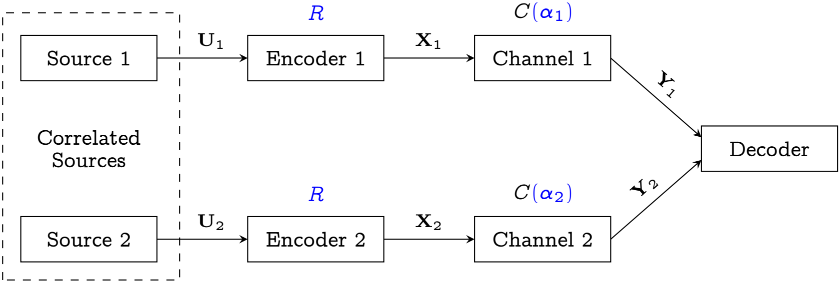

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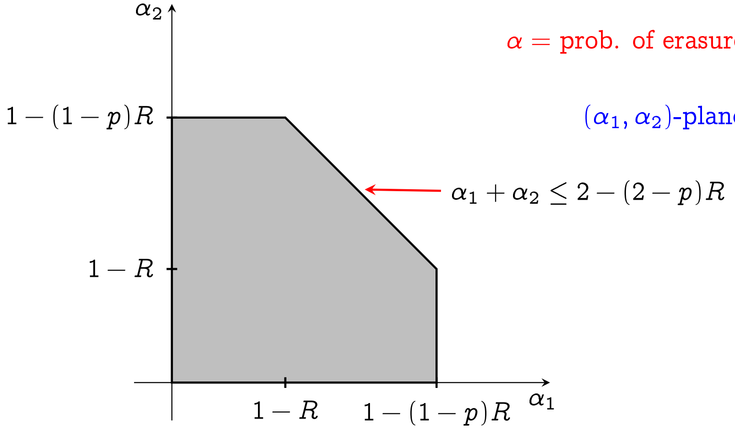
- BSC correlation over AWGN channels:
  $U_1 \sim \text{Ber}(1/2),\ Z \sim \text{Ber}(p),\ U_2 = U_1 + Z$

Slepian-Wolf (SW) Conditions



α = prob. of erasure

(α_1, α_2)-plane

erasure correlation, equal rates, and BEC noise

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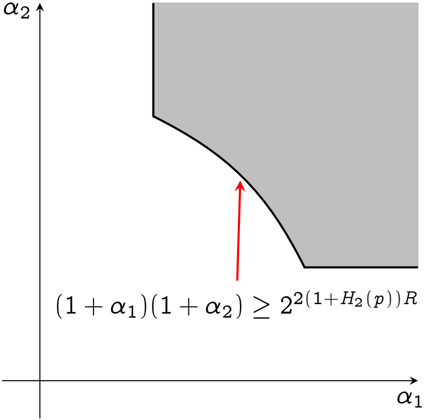

α = SNR

(α_1, α_2)-plane

BSC correlation, equal rates, and AWGN noise

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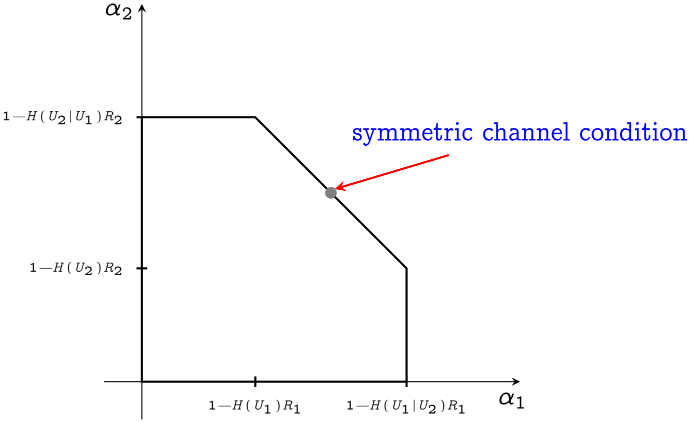

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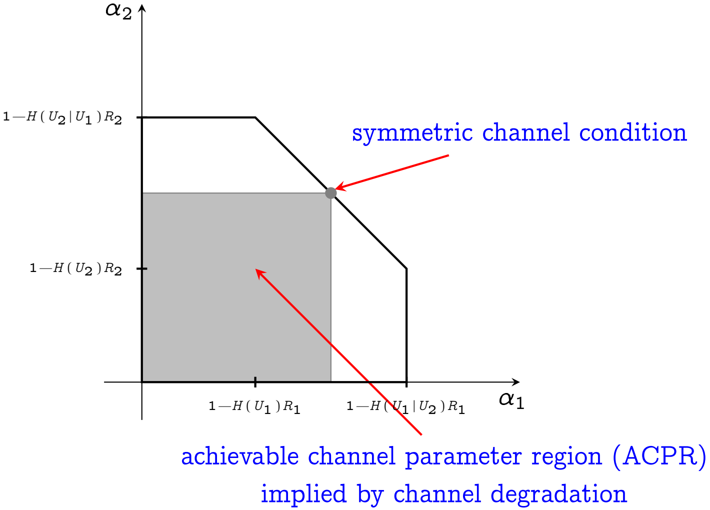

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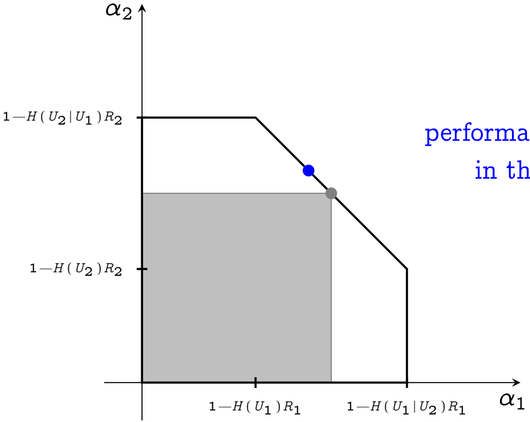
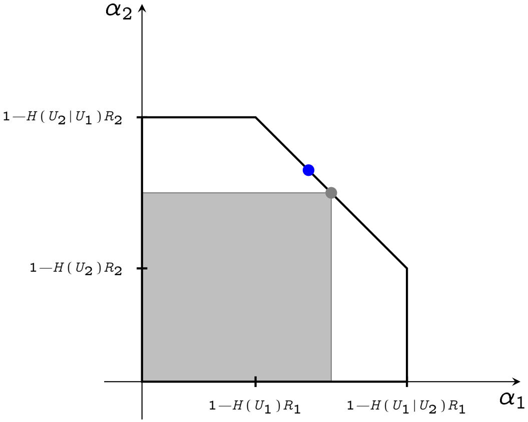


performance at another point in the feasible region

Slepian-Wolf (SW) Conditions



may be bad

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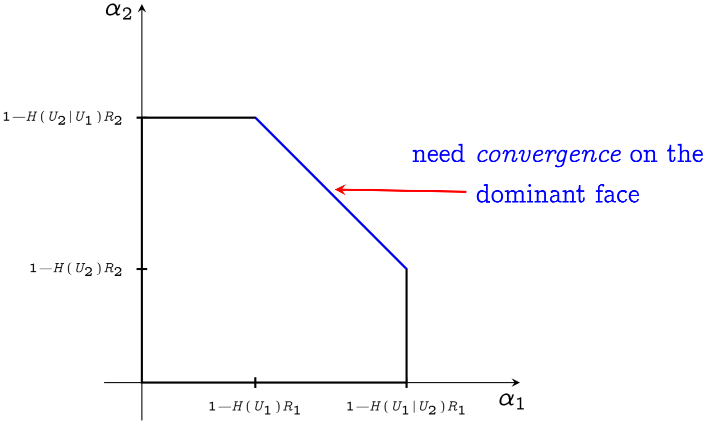

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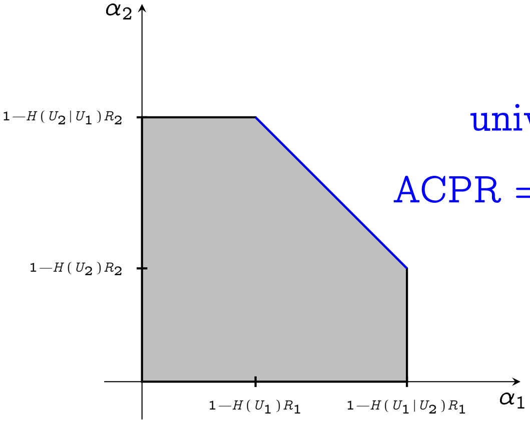

universal codes

ACPR = full SW region

SW: Motivation and Prior Work

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  - Optimized LT codes are not universal [YPN09]
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  - Systematic LDPC codes perform poorly [MFAFR10]
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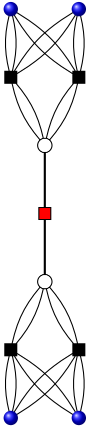
- Our Results
  - Spatially-coupled joint decoding is universal
    - Observed for BSC model and now proven for BEC model

SW: Spatially-Coupled Protograph for Joint Decoder



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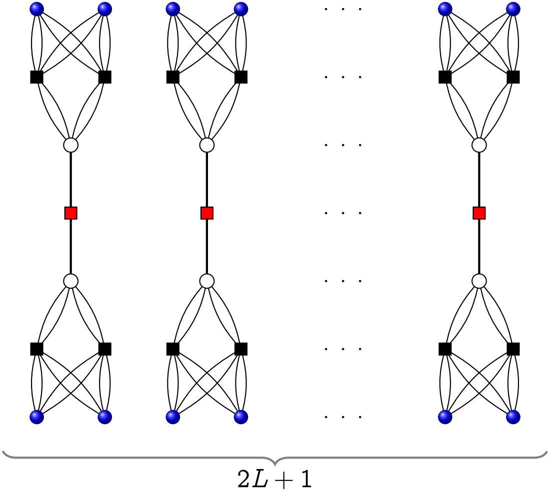

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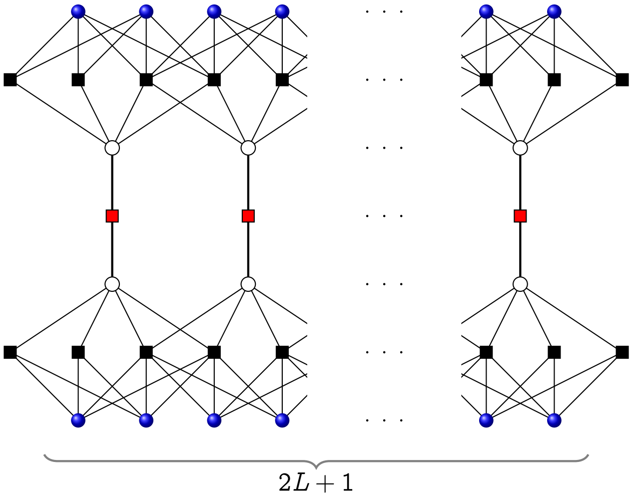
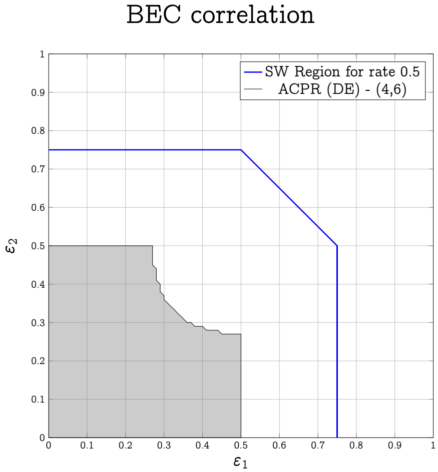
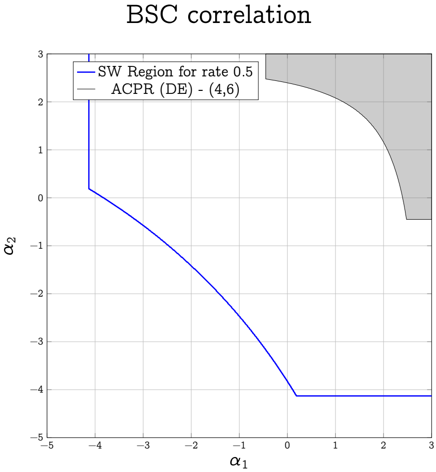


SW: DE Performance of the Joint Decoder





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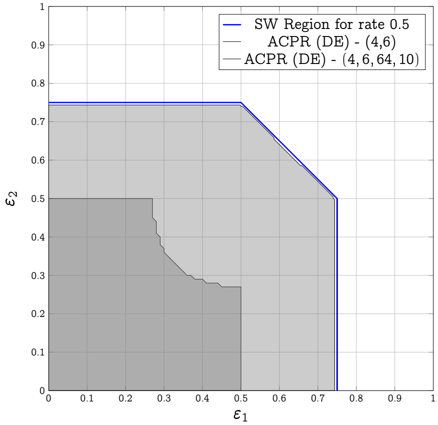
BEC correlation



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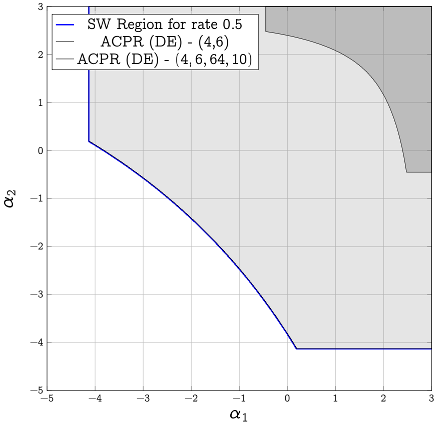

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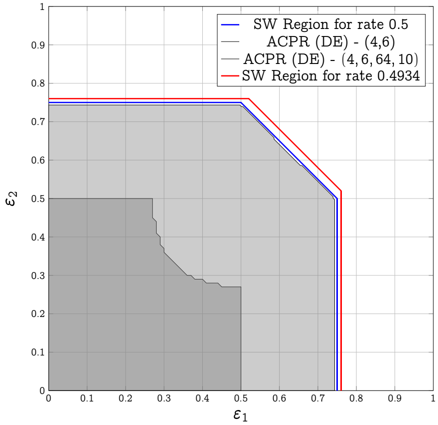

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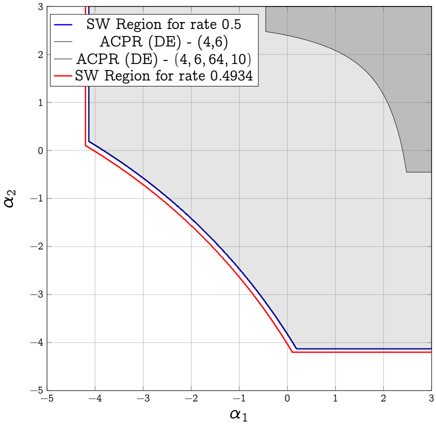

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Simple Proof of Threshold Saturation

A Simple Proof of Threshold Saturation: Outline for Scalar Recursions

$$x^{(\ell+1)} = f(g(x^{(\ell)}); \varepsilon)$$

- Statement: The threshold of coupled scalar recursions increases to an intrinsic constant defined by the scalar recursion
- Proof Outline:
  1. Define a potential function for the scalar recursion
     - Show "BP" and "MAP" thresholds can be computed from potential
  2. Derive the potential function for coupled scalar recursions
  3. Upper bound original system by modified recursion
  4. Show that, below "MAP" threshold, only fixed point is zero vector

Density Evolution for the (3,6) LDPC Ensemble

$$x^{(\ell+1)} = \varepsilon(1 - (1 - x^{(\ell)})^5)^2$$

- $f(x;\varepsilon) = \varepsilon x^2$
- $g(x) = 1 - (1 - x)^5$
- Satisfies $f(0;\varepsilon) = f(x;0) = g(0) = 0$

- For $\varepsilon < 0.4294$, no fixed point for $x \in (0, 1]$ implies $x^{(\ell)} \to 0$
- For $\varepsilon > 0.4295$, fixed point appears and $x^{(\ell)} \to x^{(\infty)} > 0.2652$

Chronicle of Threshold Saturation Proofs

- For the BEC by KRU in 2010
  - Established many properties and tools used by later approaches
- For CDMA systems with GA by TTK in 2011
  - Our use of potential functions was motivated by this paper
- For compressed sensing with GA by DJM in 2011
  - Using a vector potential function in the continuous limit
- For general scalar recursions by KRU
  - arXiv 2012 paper has a proof based on a continuous limit
- For regular codes on BMS channels by KRU in 2012
  - Threshold saturates to conjectured MAP threshold

Single-System Potential and Thresholds

- Let the potential function $U_{\mathrm{s}}(x;\varepsilon)$ of the recursion be

$$U_{\mathrm{s}}(x;\varepsilon) = \int_0^x \Big(\underbrace{z - f(g(z);\varepsilon)}_{\geq 0 \text{ if } \varepsilon < \varepsilon_{\mathrm{s}}^*} \Big) \underbrace{g'(z)}_{\geq 0} \mathrm{d}z$$

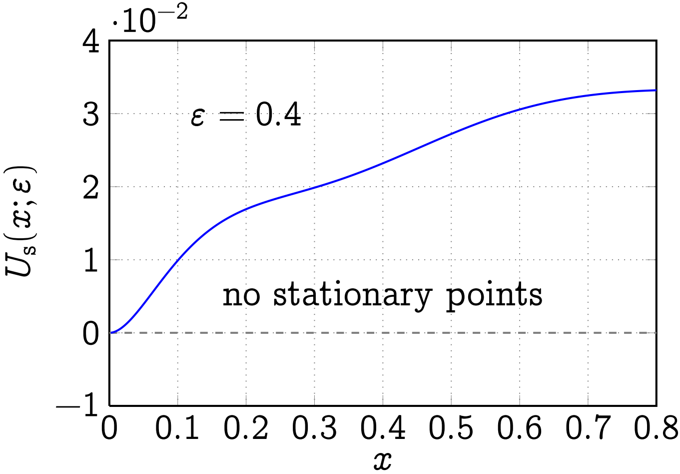
$$= xg(x) - G(x) - F(g(x);\varepsilon),$$

where $F(x;\varepsilon) = \int_0^x f(z;\varepsilon)\mathrm{d}z$ and $G(x) = \int_0^x g(z)\mathrm{d}z$.

- Single system "BP" threshold: $f(g(z);\varepsilon) < z$ for $\varepsilon < \varepsilon_{\mathrm{s}}^*$
- The potential threshold ε^* is the "MAP" threshold (i.e., the supremum of all ε such that $U_{\mathrm{s}}(x;\varepsilon) \geq 0$ for all x)

Potential Function for the (3,6) LDPC Ensemble

$$U_s(x;\varepsilon) = x(1-(1-x)^5) - \frac{6x-1+(1-x)^6}{6} - \varepsilon\frac{(1-(1-x)^5)^3}{3}$$



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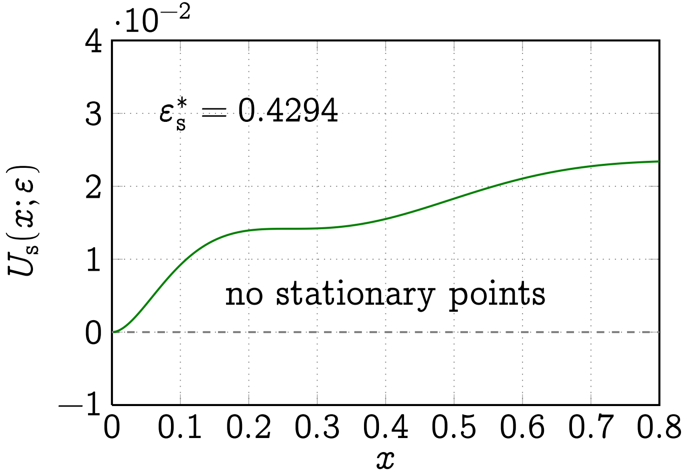

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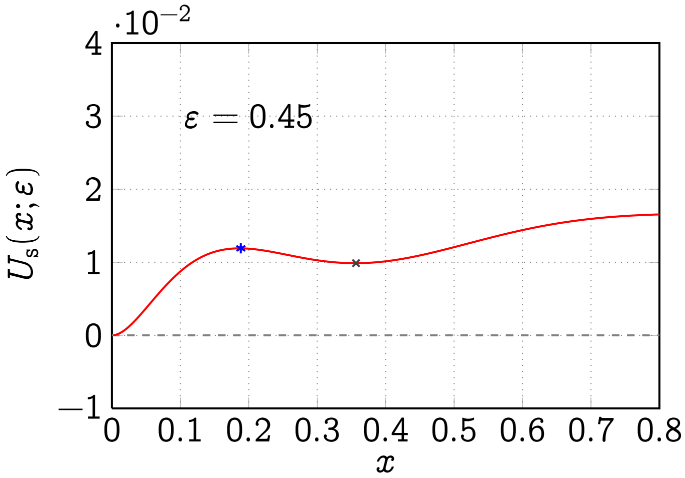

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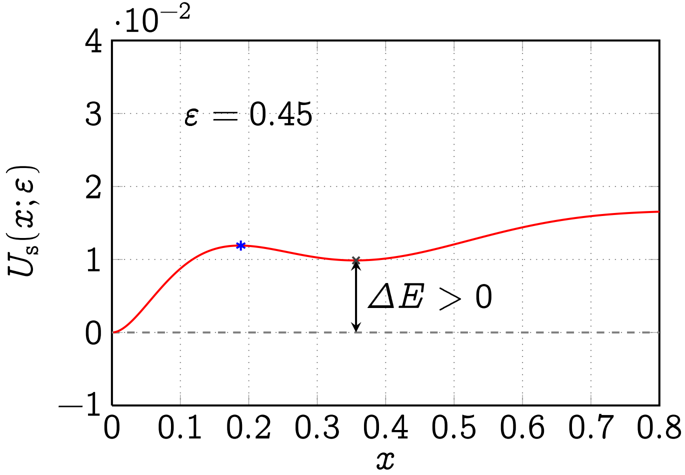

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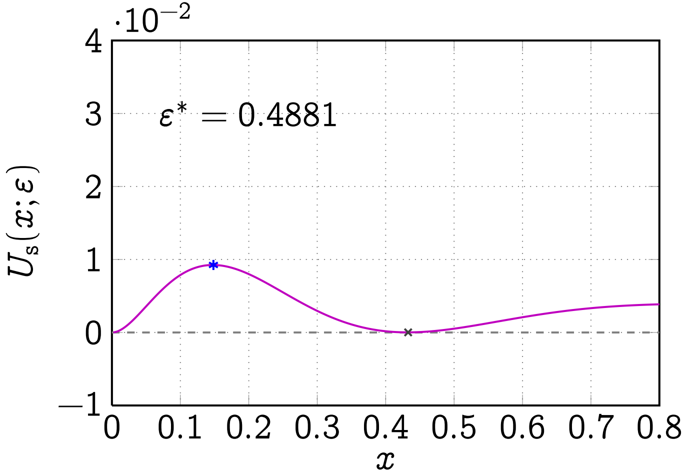

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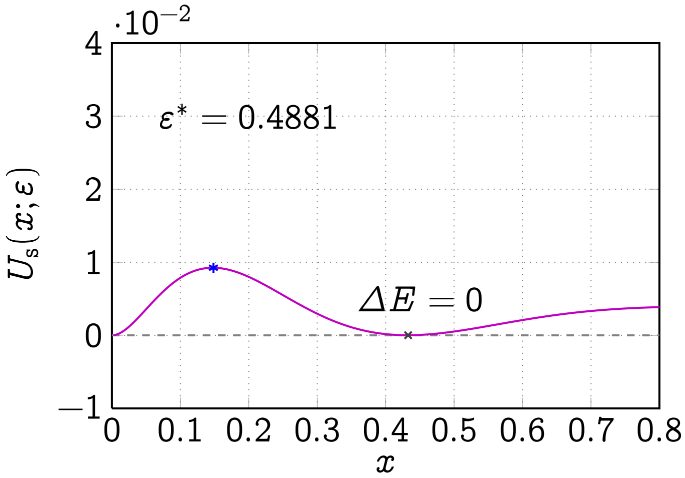

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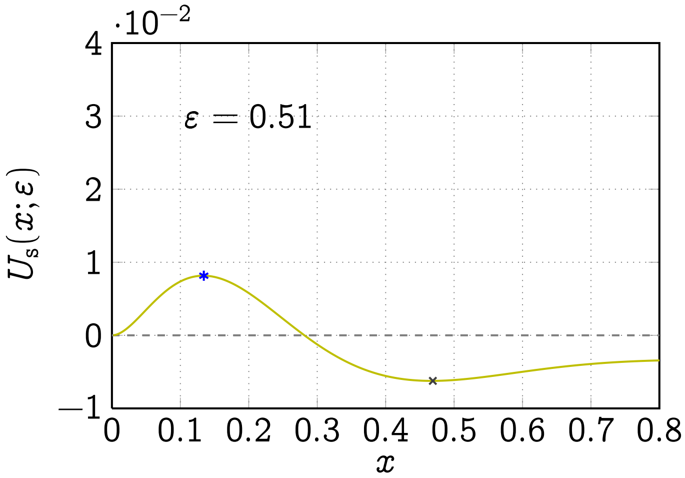

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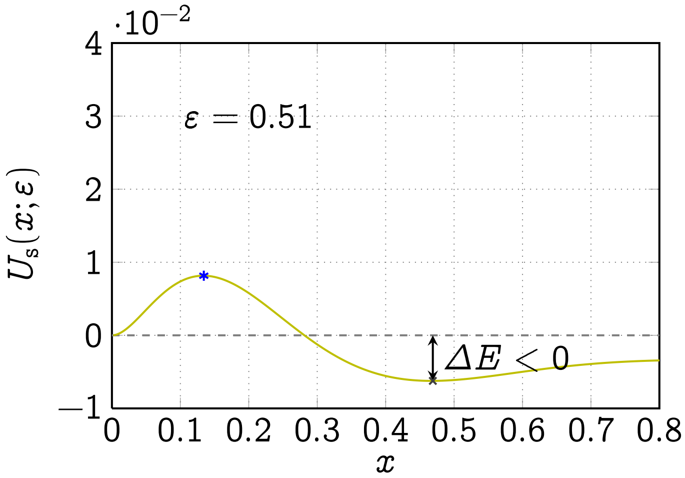
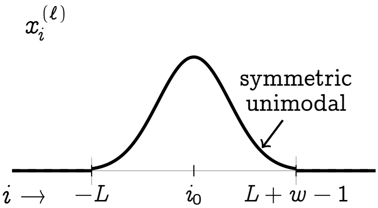


Coupled Recursions

$$x_i^{(\ell+1)} = \frac{1}{w} \sum_{k=0}^{w-1} f \left(\frac{1}{w} \sum_{j=0}^{w-1} g \left(x_{i+j-k}^{(\ell)} \right) ; \varepsilon_{i-k} \right)$$



$$x_i^{(0)} = 1, i \in \{-L, \cdots, L+w-1\}$$

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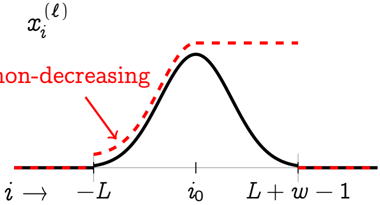
Modified recursion upper bounds original recursion



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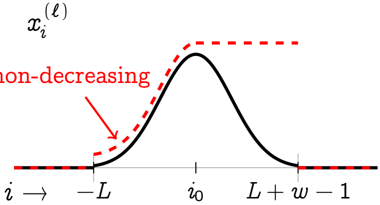
Original update equals modified update only for $-L \leq i \leq i_0$



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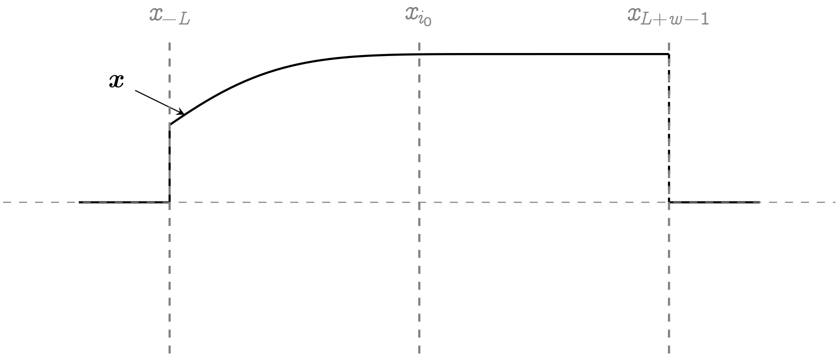
$$U_{\mathsf{c}}(\boldsymbol{x};\varepsilon) = \int_C \boldsymbol{g}'(\boldsymbol{z})(\boldsymbol{z} - \boldsymbol{A}^{\intercal}\boldsymbol{f}(\boldsymbol{A}\boldsymbol{g}(\boldsymbol{z});\varepsilon)) \cdot \mathrm{d}\boldsymbol{z}$$
$$= \boldsymbol{g}(\boldsymbol{x})^{\intercal}\boldsymbol{x} - G(\boldsymbol{x}) - F(\boldsymbol{A}\boldsymbol{g}(\boldsymbol{x});\varepsilon)$$

- where $G(\boldsymbol{x}) = \sum_i G(x_i)$ and $F(\boldsymbol{x};\varepsilon) = \sum_i F(x_i;\varepsilon)$

- $U_{\mathsf{c}}'(\boldsymbol{x};\varepsilon) = \boldsymbol{g}'(\boldsymbol{z})(\boldsymbol{z} - \boldsymbol{A}^{\intercal}\boldsymbol{f}(\boldsymbol{A}\boldsymbol{g}(\boldsymbol{z});\varepsilon))$ equals $\mathbf{0}$ at any fixed point

Properties of Shift Matrix

Down-shift matrix $\boldsymbol{S}$ = right shift in picture



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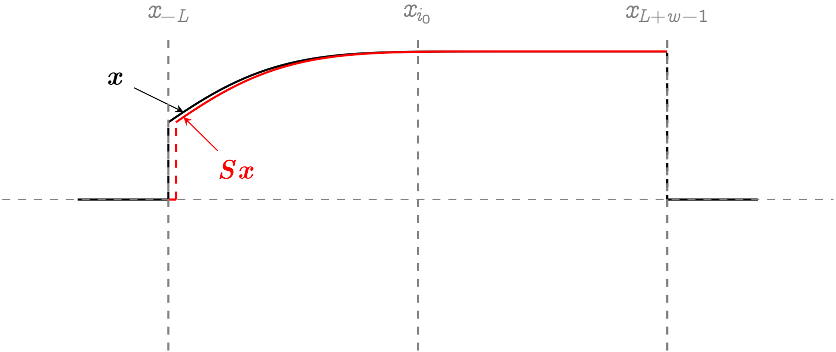

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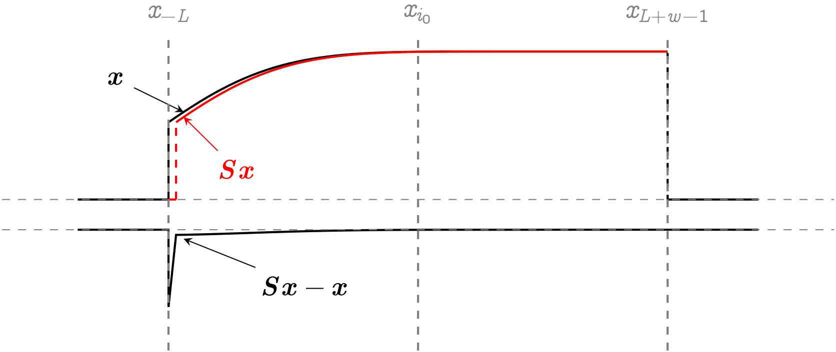

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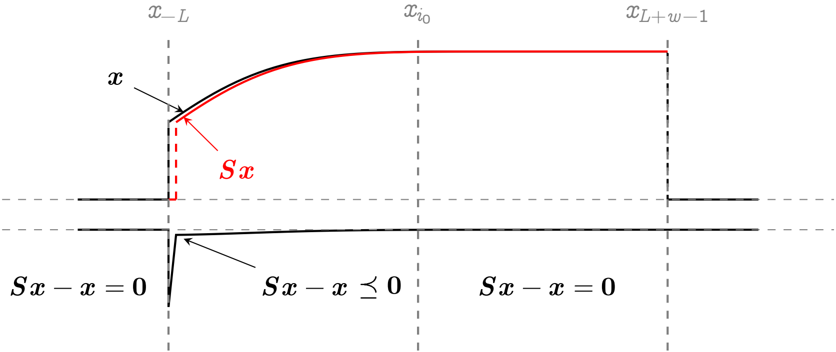

$$\|\boldsymbol{Sx} - \boldsymbol{x}\|_\infty \leq \frac{1}{w} \qquad \|\boldsymbol{Sx} - \boldsymbol{x}\|_1 = x_{i_0} \leq 1$$

Sketch of Proof (1)

- Fix $\varepsilon < \varepsilon^*$ and choose $w > K_{f,g} / \left(2\Delta E(\varepsilon)\right)$
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&\leq \tfrac{1}{2}\left\|\boldsymbol{S}\boldsymbol{x}-\boldsymbol{x}\right\|_1 \left\|U_{\mathsf{c}}''(\boldsymbol{z}(t_0);\varepsilon)\right\|_\infty \left\|\boldsymbol{S}\boldsymbol{x}-\boldsymbol{x}\right\|_\infty \\
&\leq \tfrac{1}{2}\cdot 1\cdot K_{f,g}\cdot \tfrac{1}{w}
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Therefore, $\mathbf{0}$ is the only f.p. of the modified recursion and also the only f.p. of the original recursion

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- For many multiuser problems,
  - Optimized LDPC codes are not universal
  - But suboptimal decoding is the main problem
  - Spatially-coupled joint decoders appear to be universal
    - Observed in general and now proven for some erasure models

Thanks for your attention