# Reed-Muller Codes Achieve Capacity on Erasure Channels

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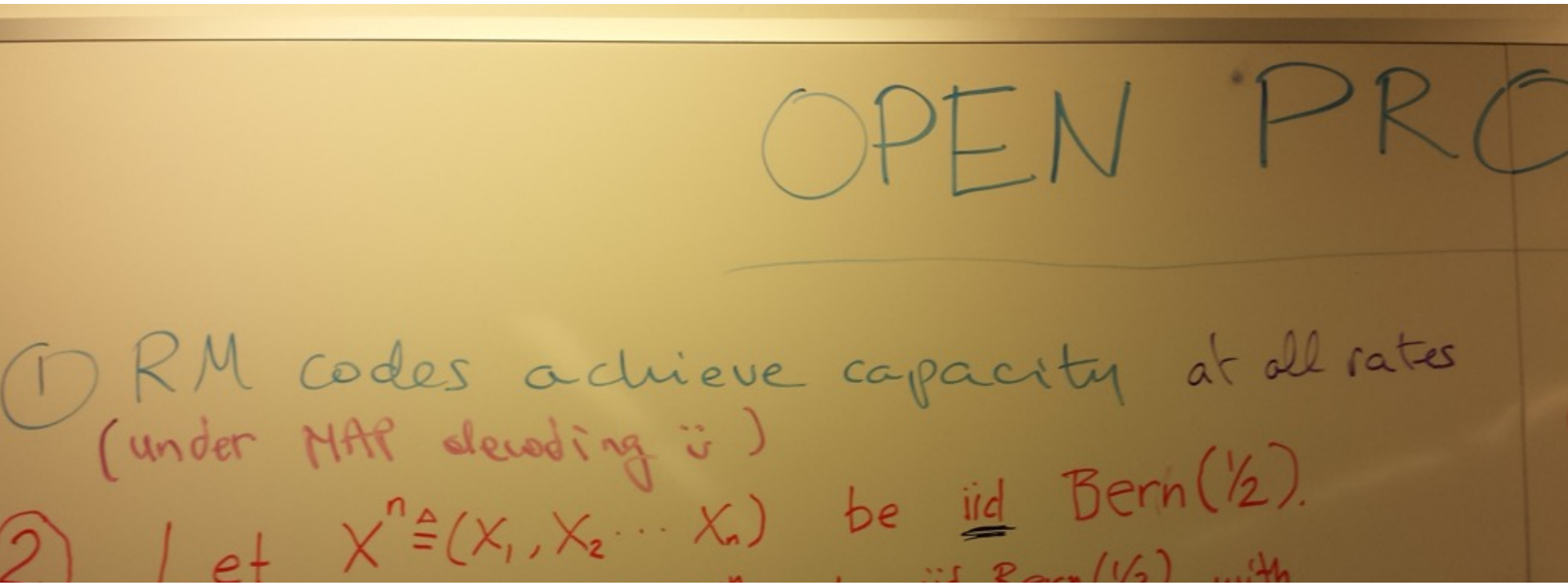
OPEN PRO
(1) RM codes achieve capacity at all rates
(under MAP decoding ü)
2) Let $X^n \triangleq (X_1, X_2 \cdots X_n)$ be iid Bern(½).


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# Reed-Muller (RM) Codes (I)

- Codes by Muller, decoder by Reed, both in 1954
  - Multivariate polynomial evaluation codes over the binary field:

$$\mathrm{RM}(v,n) \triangleq \left\{ \underline{c} \in \mathbb{F}_2^{2^n} \,\middle|\, c_{\underline{x}} = f(\underline{x}), \underline{x} \in \mathbb{F}_2^n, f \in \mathbb{F}_2[x_1,\ldots,x_n], \deg(f) \leq v \right\}$$

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- First known conjecture in print by Dumer and Farrell in 1994
  - They show BCH codes achieve capacity on BEC as rate $\to 1$
  - Open problem stated for Reed-Muller codes with constant rate

# Reed-Muller (RM) Codes (II)

- Closely related to polar codes
  - From Hadamard matrix, one choice of rows generates Reed-Muller and some other polar codes

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- In 2014, Abbe-Shpilka-Wigderson showed capacity achieving for rates $\to 0, 1$ (erasures) and rates $\to 0$ (errors)
- Do Reed-Muller codes achieve capacity for constant rates?

# A Brief Note on Symmetry

- Consider a set $\mathcal{C} \subset \mathcal{X}^N$ of length-$N$ vectors
  - The permutation group of $\mathcal{C}$ is defined to be

$\mathcal{G} \triangleq \{\pi \in S_N \,|\, \pi(\underline{c}) \in \mathcal{C} \;\forall\; \underline{c} \in \mathcal{C}\}$ where $\underline{d} = \pi(\underline{c}) \Leftrightarrow d_{\pi(i)} = c_i$

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- For example,

$$\mathcal{C} = \left\{ \begin{array}{cccc} (0 & 0 & 0 & 0) \\ (0 & 0 & 1 & 1) \\ (1 & 1 & 0 & 0) \\ (1 & 1 & 1 & 1) \end{array} \right\} \quad \Longrightarrow \quad \mathcal{G} = \left\langle \left( \begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{smallmatrix} \right) \left( \begin{smallmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{smallmatrix} \right) \right\rangle$$

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  - A permutation group $\mathcal{G}$ is transitive if, for all $i, j \in [N]$, there exists $\pi \in \mathcal{G}$ such that $\pi(i) = j$
  - $\mathcal{G}$ is doubly transitive if, for any $i, j, k, l \in [N]$ with $i \neq j$ and $k \neq l$, there exists $\pi \in \mathcal{G}$ such that $\pi(i) = k$ and $\pi(j) = l$.
- For example, $\mathcal{C}$ is transitive but not doubly transitive

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# Capacity via Symmetry for Erasure Channels

Theorem: Let $\{\mathcal{C}_n\}$ be a sequence of binary linear codes with

- blocklengths $N_n \to \infty$ and rates $r_n \to r \in (0, 1)$ where
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## Important Consequences and Extensions

- Permutation groups of RM codes are doubly transitive [KLP]

  $\implies$ Reed-Muller codes achieve capacity
- Extends to linear affine-invariant codes over $\mathbb{F}_q$
- Primitive narrow-sense BCH codes achieve capacity
- Extends to block-MAP decoding for Reed-Muller and BCH

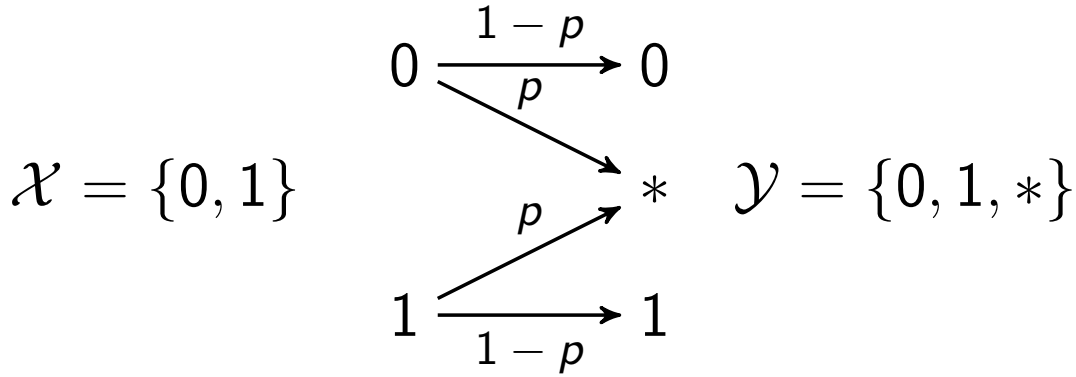
# Few Remarks

- Scope of the work
  - Linear Codes, Erasure Channels, MAP Decoding
- Three Main Ingredients
  - EXIT functions (from iterative decoding)
  - Monotone boolean functions (from computer science)
  - Automorphism/Permutation groups (from algebraic coding)

# Proof

# Basic Setup

- Binary linear code $\mathcal{C} \subset \{0,1\}^N$ is a $K$-dim. subspace of $\mathbb{F}_2^N$
- Binary Erasure Channel, parametrized by $p$




- $\underline{X} = (X_1, \ldots, X_N) \longleftrightarrow$ uniform codeword from $\mathcal{C}$
- $\underline{Y} = (Y_1, \ldots, Y_N) \longleftrightarrow$ received sequence from $\underline{X}$

# MAP Decoding on Erasure Channels

## Set of Consistent Codewords

$$\mathcal{C}(\underline{y}) = \{\underline{x} \in \mathcal{C} \quad | \quad x_i = y_i \quad \text{when } y_i \neq *\}$$

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- Otherwise
  - Half of codewords in $\mathcal{C}(\underline{y})$ have $x_i = 0$ and half have $x_i = 1$
  - uniform codeword $\longleftrightarrow$ uniform posterior
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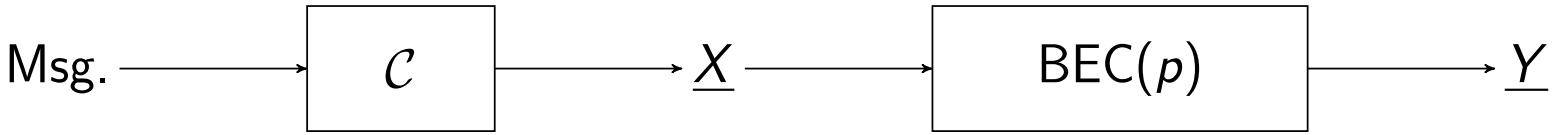
  - uniform codeword $\longleftrightarrow$ uniform posterior
  - $H(X_i|\underline{Y} = \underline{y}) = 1$
- $H(X_i|\underline{Y} = \underline{y})$ is either 0 or 1 (Boolean)

# First Ingredient: MAP EXIT Functions




EXtrinsic Information Transfer Function

## Definition

(Bit-$i$ MAP Erasure Prob.) $P_{b,i} = H(X_i \mid \underline{Y})$

(MAP Erasure Probability) $P_b = \frac{1}{N}\sum_{i=1}^{N} P_{b,i}$

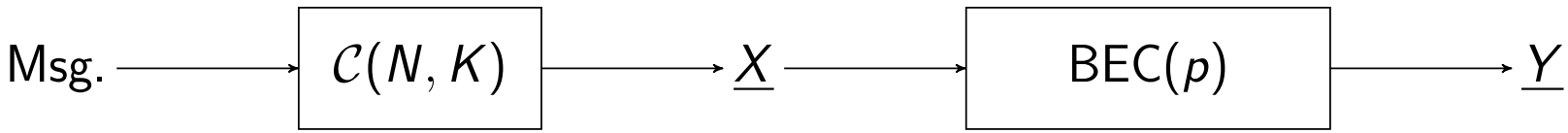
(Bit-$i$ EXIT Function) $h_i(p) = H(X_i \mid \underline{Y}_{\sim i})$

(Average EXIT Function) $h(p) = \frac{1}{N}\sum_{i=1}^{N} h_i(p)$

- Note: $\underline{Y}_{\sim i} = (Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_N)$
- Closely related to each other:

$$P_{b,i}(p) = p h_i(p) \qquad P_b(p) = p h(p)$$

# EXIT Functions and the Area Theorem

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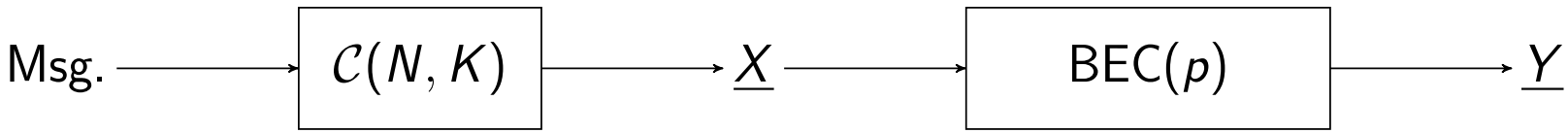


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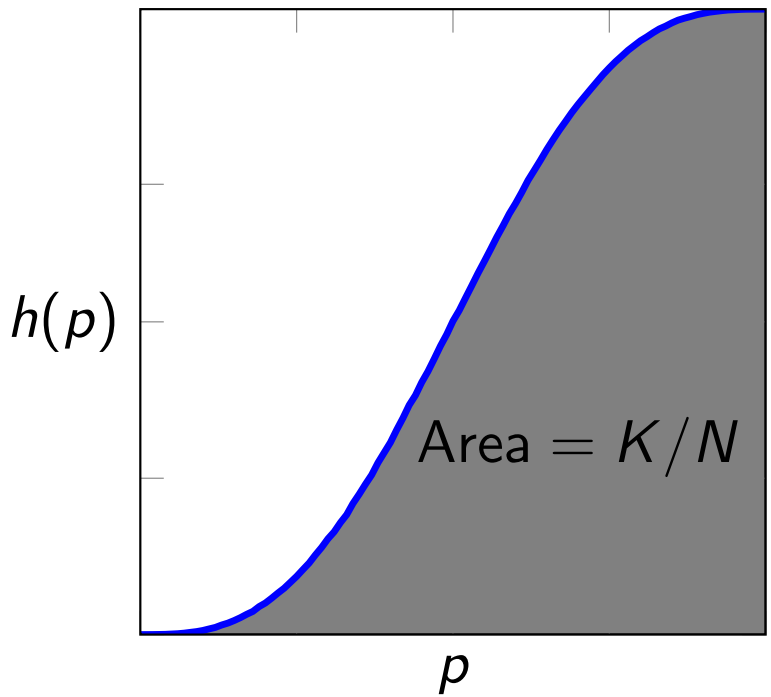
## Area Theorem

$$\int_0^1 h(p)dp = K/N$$

- Conservation Principle by Ashikhmin, ten Brink, and Kramer
- Not satisfied by $P_b$



# Capacity and EXIT Functions

A sequence of codes $\{\mathcal{C}_n\}$ with rates $r_n \to r$

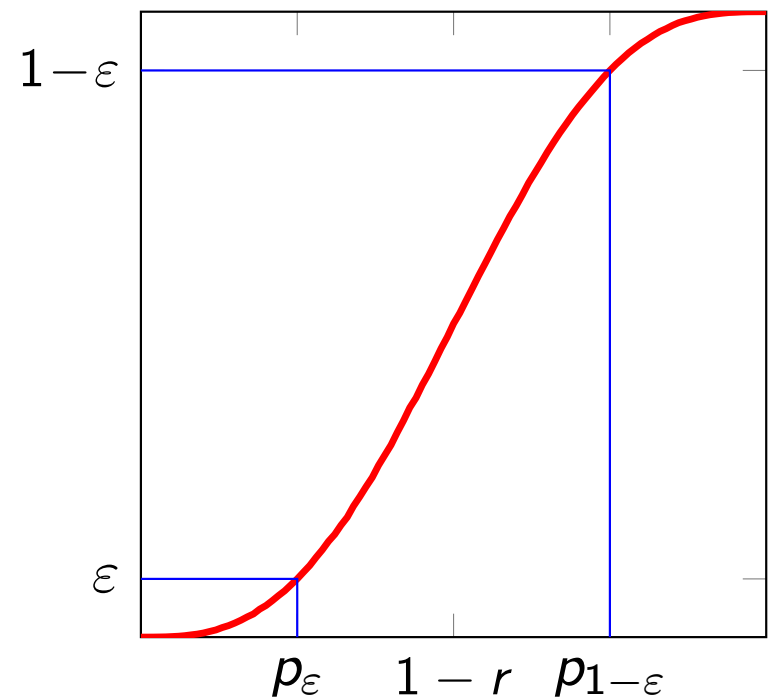
is capacity-achieving if $P_b^{(n)}(p) \to 0$ for all $p < 1 - r$.

The following are equivalent:

- $\{\mathcal{C}_n\}$ is capacity achieving
- $h^{(n)}(p) \to \begin{cases} 0, & \text{if } p < 1 - r, \\ 1, & \text{if } p > 1 - r. \end{cases}$
- Transition width vanishes

  For all $\varepsilon > 0$, $p_{1-\varepsilon}^{(n)} - p_{\varepsilon}^{(n)} \to 0$,

  where $p_\epsilon \triangleq h^{-1}(\epsilon)$



# Second Ingredient: Monotone Boolean Functions

## Definition

A boolean function $f_i : \{0,1\}^{N-1} \to \{0,1\}$ is uniquely defined by the set $\Omega_i \triangleq \{\underline{z} \in \{0,1\}^{N-1} \mid f_i(\underline{z}) = 1\}$ and called monotone if

$$\underline{z} \leq \underline{z}' \text{ implies } f_i(\underline{z}) \leq f_i(\underline{z}').$$

Consider the expectation $\mathbb{E}[f_i(\underline{Z})]$ when $\underline{Z}$ is an i.i.d. $B(p)$ vector

Let set of erasure patterns that prevent recovery of $X_i$ from $\underline{Y}_{\sim i}$ be

$$\Omega_i \triangleq \{\underline{z}_{\sim i} \in \{0,1\}^{N-1} \mid \exists\, \underline{x} \in \mathcal{C}, x_i = 1, \underline{x}_{\sim i} \leq \underline{z}_{\sim i}\},$$

Since adding erasures cannot help recover $X_i$, $f_i$ is monotone and

$$h_i(p) = \mathbb{P}(\underline{Z} \in \Omega_i) = \mathbb{E}[f_i(\underline{Z})]$$

# Symmetric Montone Boolean Functions

$f_i$ "symmetric" $\Leftrightarrow$ permutation group of $\Omega_i$ is transitive

- In this case, $h_i(p)$ has a sharp transition!
  ( Friedgut-Kalai, Talagrand, Bourgain-Kahn-Kalai-Linial )

$$p_{1-\varepsilon} - p_\varepsilon \leq 2C \frac{\log \frac{1}{\varepsilon}}{\log N}, \qquad p_{1-\varepsilon} - p_\varepsilon \to 0.$$

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Avg. EXIT Function $h$, not $h_i$, satisfies Area Theorem

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**Symmetry to the rescue!**

# Third Ingredient: Group Symmetry

The permutation group $\mathcal{G}$ of code $\mathcal{C}$ is defined to be

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## Proposition

- If $\mathcal{G}$ is transitive, then

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$$h_i(p) = h_j(p) = h(p) \quad \text{for all } 0 \leq p \leq 1$$

- If $\mathcal{G}$ is doubly transitive, then

$f_i$ is "symmetric" (i.e., permutation group of $\Omega_i$ is transitive)
and $h_i$ has a sharp transition

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Under double transitivity: $h_i = h$ and $f_i$ is symmetric

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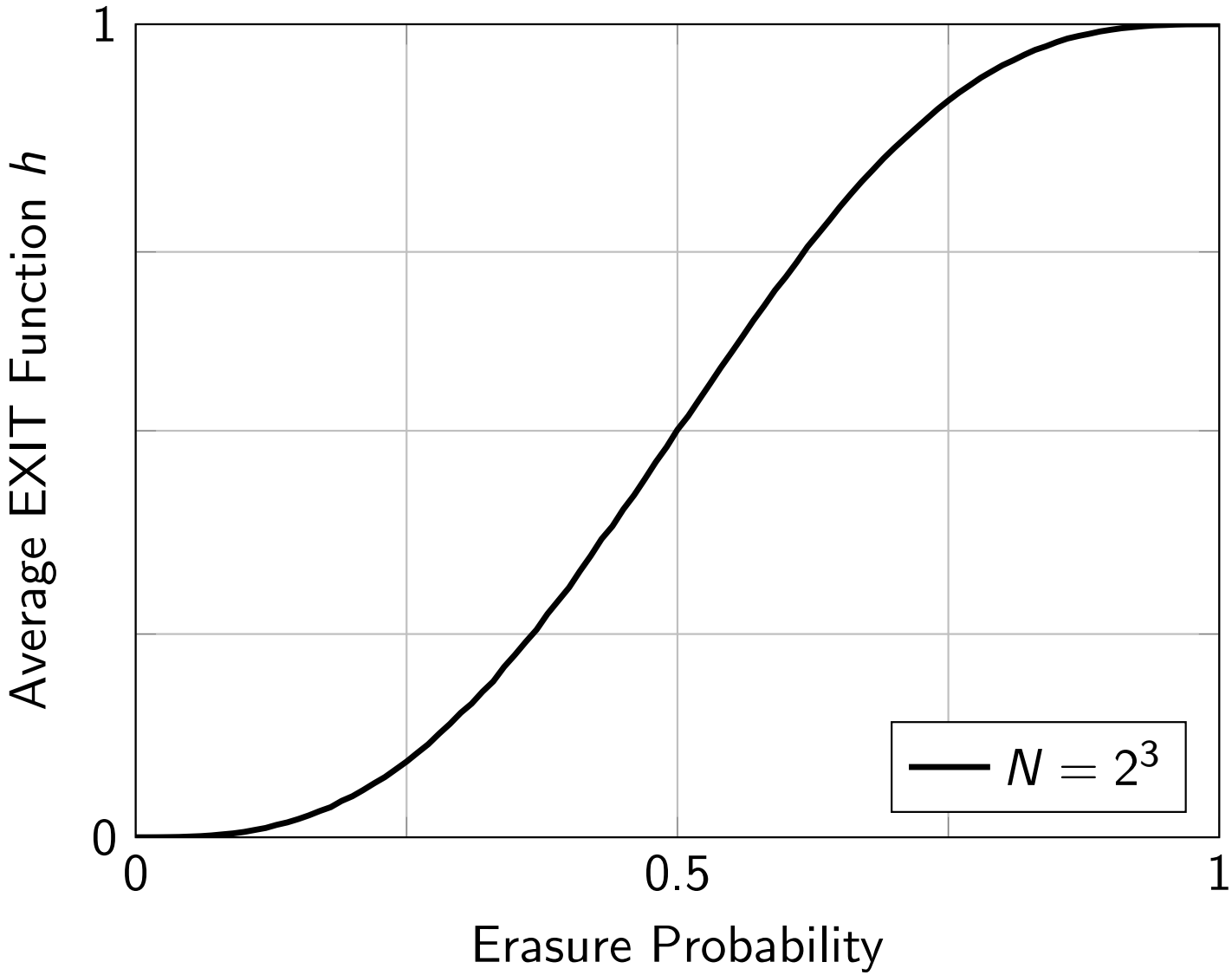
Symmetric Monotone Boolean Functions
Exhibit Sharp $0-1$ Transitions

Avg. EXIT Function $h$ has a sharp $0-1$ transition
Area theorem implies transition at $1-r$

# Rate-1/2 Reed-Muller Codes



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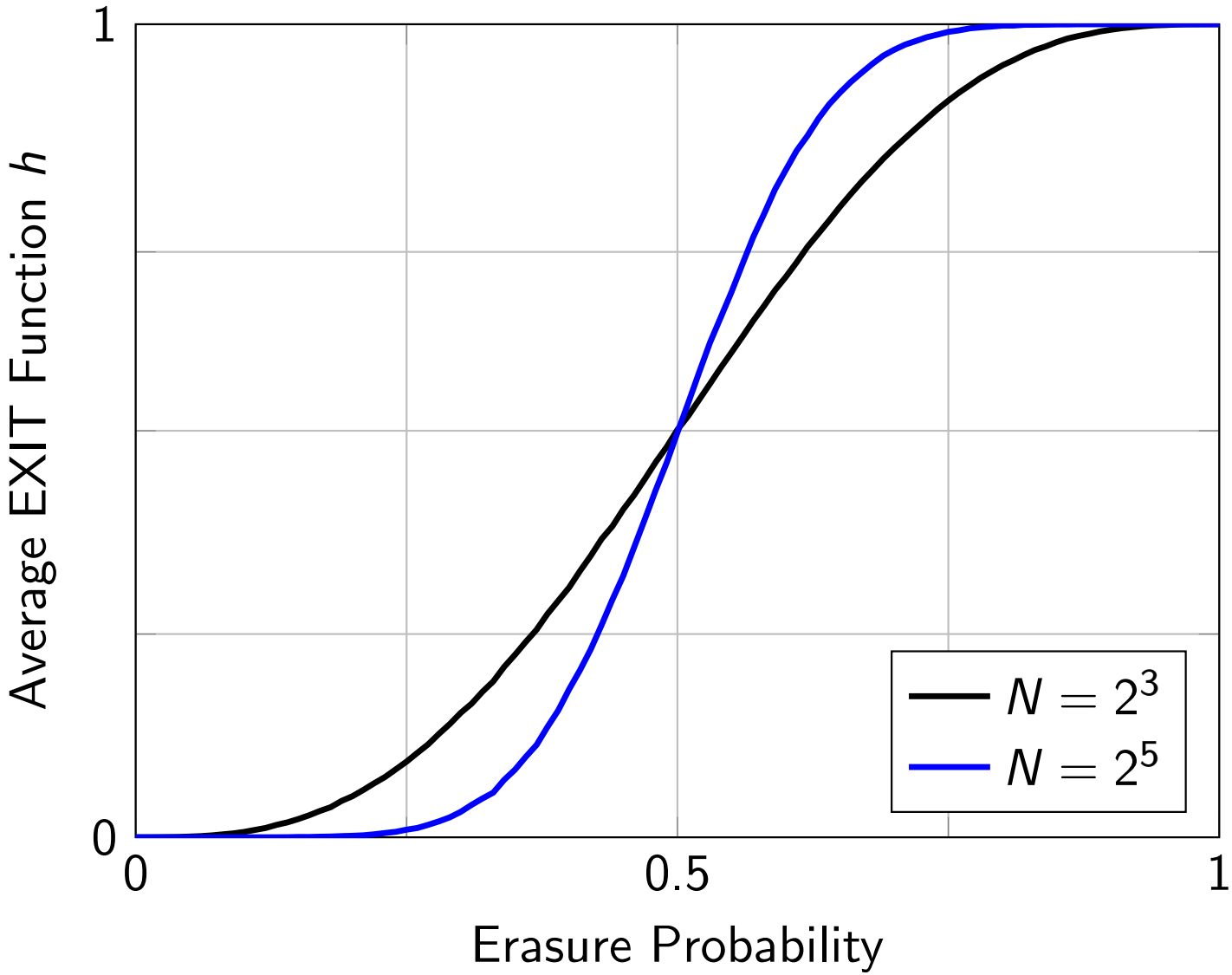

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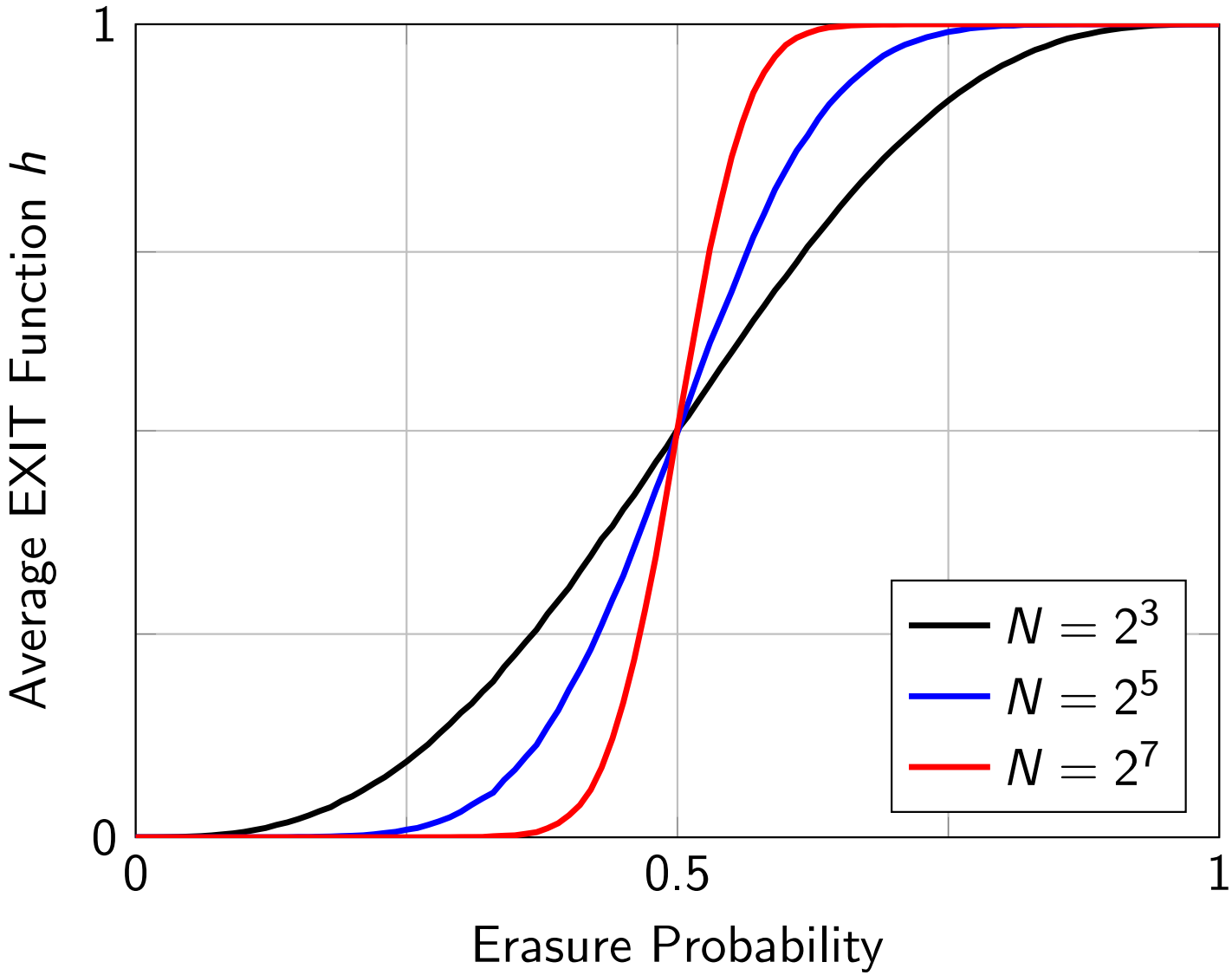

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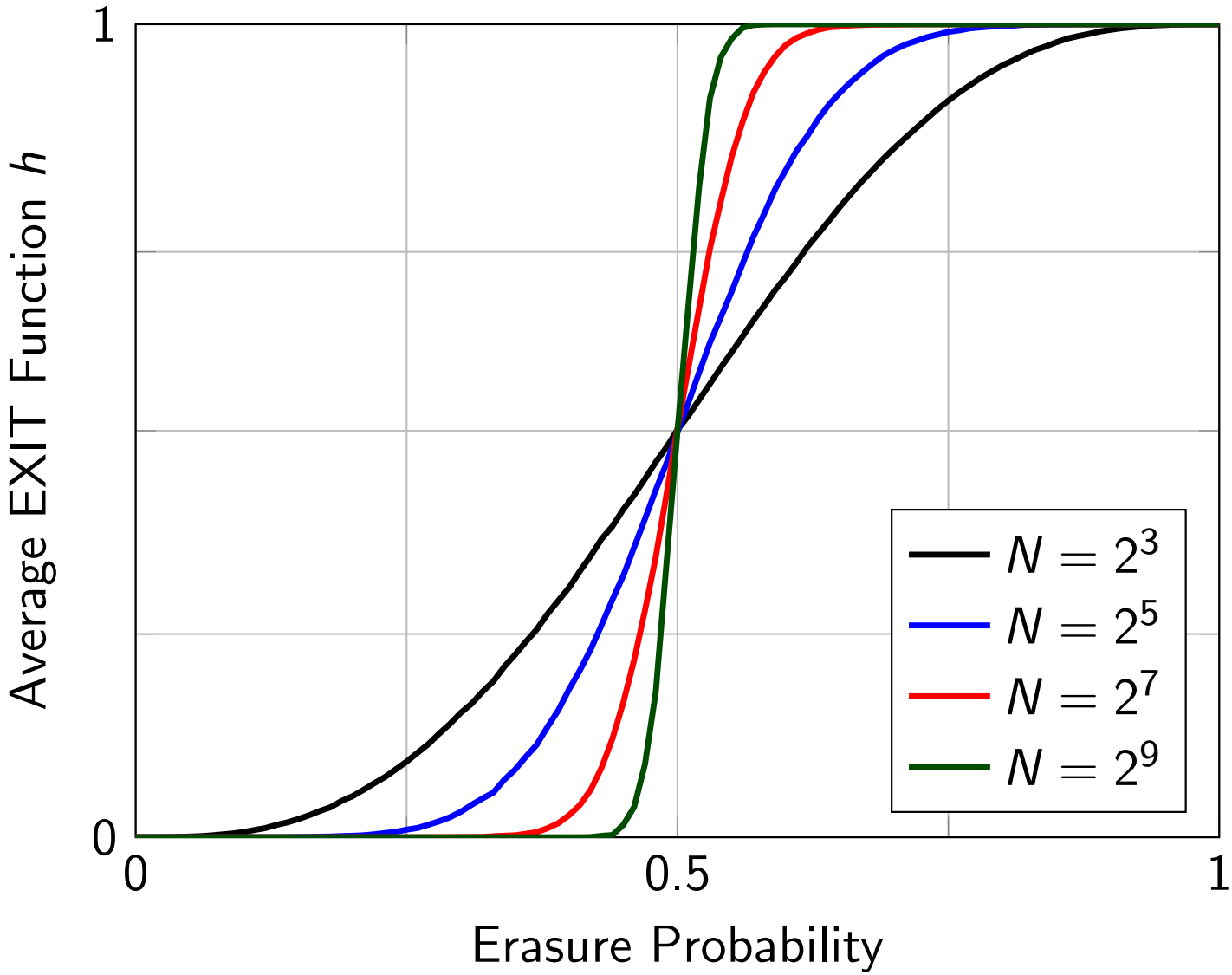

# Capacity via Symmetry

## Generality

- How general is this phenomenon?
- Proof heavily exploits MAP decoding on erasure channels
- Abbe et al. have shown for BSC when rate $\to 0$ [ASW]
- Extends to block-error rate for RM and BCH codes [BK]

## Open Questions

- Extension to general BMS channels
  - All ingredients except the sharp transition generalize naturally
- Practical decoders that achieve capacity for non-trivial rates
- Extension for BEC to rates converging to 0 or 1
  - Friedgut extended sharp threshold result to $p \to 0$ for $K$-SAT

# Thank You!