Upper Bounds on the MAP Threshold of Iterative Decoding Systems with Erasure Noise

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Outline

1. Introduction
    - Iterative Decoding Systems
    - EXIT Functions and the Area Theorem
    - Bounding the MAP Threshold
2. MAP Threshold Bounds for IRA/ARA Ensembles
    - Systematic Irregular Repeat-Accumulate (SIRA) Codes
    - Accumulate Repeat-Accumulate (ARA) Codes
    - Tightness of the Upper Bound
3. Joint Decoding of LDPC Codes and Channels with Memory
    - MAP Threshold Bounds for Joint Decoding
4. Conclusions

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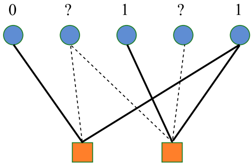
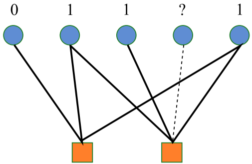
Sparse Graph Codes

- Ensembles defined via structured random parity-check matrix
  - Low-Density Parity-Check (LDPC), Repeat Accumulate (RA)
  - Irregular RA (IRA), Accumulate RA (ARA), etc...
  - Ensembles defined by a degree distribution (d.d.) $(\lambda(x), \rho(x))$
- Belief Propagation (BP) Decoding
  - Density evolution (DE) used to compute a noise threshold where bit error rate vanishes (w.h.p. as $n \to \infty$) if the channel is better than the threshold
  - Optimized degree distributions achieve good performance at rates near capacity

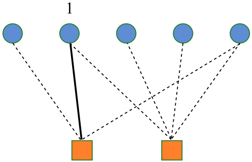
Belief Propagation Decoding



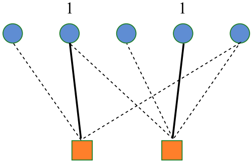
bit-to-check: iteration 1



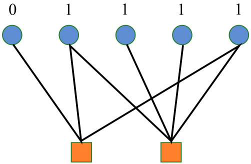
bit-to-check: iteration 2



check-to-bit: iteration 1



check-to-bit: iteration 2



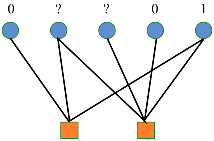
bit-to-check: iteration 3

Maximum A Posteriori (MAP) Envy

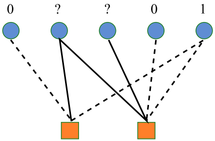
- BP decoding is good, but how much better is MAP decoding?
  - Although BP is optimal on a tree, it is suboptimal in general
  - Méasson, Montanari, and Urbanke (MMU) addressed this question by focusing on the Area Theorem for EXtrinsic Information Transfer (EXIT) charts
  - Results were given for LDPC and Turbo codes

- The MMU approach is quite general
  - Upper bounds on the MAP threshold are computable for ensembles which admit DE analysis
  - Lower bounds on the MAP threshold (for the BEC) are computable for ensembles which allow weight enumerator (WE) analysis after the peeling decoder gets stuck

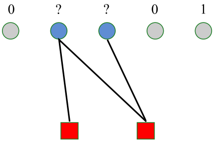
Peeling Decoding



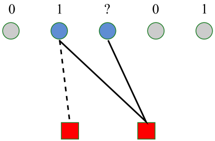
initial graph and received word



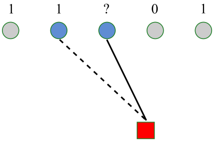
initialization (a): channel values



initialization (b): merge values into checks



step 1a: Degree-1 check gives value



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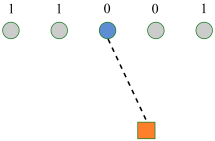

step 2a: Degree-1 check gives value

Our Results

- MAP threshold bounds for IRA and ARA codes on the BEC
  - Applies the MMU approach
  - Upper bound based on IRA/ARA DE equations for EXIT chart
  - Lower bound based on graph reduction to LDPC code followed by peeling decoder and WE analysis
- MAP threshold bounds for an LDPC code over an ISI channel
  - Follows MMU approach with slight change in EXIT theorem
  - Upper bound via DE analysis of the joint iterative decoder of code and channel (e.g., turbo equalization)
  - Lower bound is a work in progress...
    - Modified graph reduction for the channel decoder, but...

How is the MAP Threshold Useful?

- Separates rate loss from capacity in two parts
    - MAP threshold vs Capacity gives loss due to code
    - BP threshold vs MAP threshold gives iterative decoding loss
    - Maxwell $O\left(n^2\right)$ decoding for BEC achieves MAP threshold

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MAP EXIT Function

Definition (MAP EXIT Function)

Let C be a length-n binary code defined by $p_{X_1^n}(x_1^n)$. Codeword X_1^n is chosen according to $p_{X_1^n}(x_1^n)$ and Y_1^n is the result of transmitting X_1^n over a BEC(ϵ). Then, the *MAP EXIT function* is

$$h_C^{MAP}(\epsilon) \triangleq \frac{1}{n}\sum_{i=1}^{n} H\left(X_i | Y_1^n(\epsilon) \setminus Y_i(\epsilon)\right).$$

Theorem (Area Theorem)

$$\frac{1}{n}H\left(X_1^n | Y_1^n(\epsilon)\right) = \int_0^{\epsilon} h_C^{MAP}(\delta)\, d\delta$$

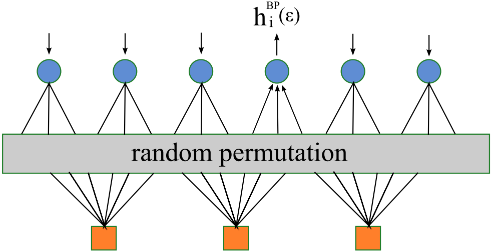
BP EXIT function

Definition (BP EXIT Function)

The *BP EXIT function* of a length-n code C is given by

$$h_{C,l}^{BP}(\epsilon) \triangleq \frac{1}{n}\sum_{i=1}^{n} h_i^{BP}(\epsilon),$$

where $h_i^{BP}(\epsilon)$ is the entropy of X_i given the extrinsic message from X_i to the channel (function of $Y_1^n \backslash Y_i$) after l iterations of decoding.



Properties of EXIT Functions

1. $0 \le h_C^{MAP}(\epsilon) \le h_{C,l}^{BP}(\epsilon) \le 1$ and both are non-decreasing
2. Code rate of C is $R_C = \frac{1}{n} H(X_1^n) = \int_0^1 h_C^{MAP}(\epsilon) d\epsilon$
3. For an ensemble sequence $\mathcal{C}_n$ of increasing length
   (a) Asymptotic MAP EXIT: $h_{\mathcal{C}}^{MAP}(\epsilon) \triangleq \limsup_{n\to\infty} E_{\mathcal{C}_n}\left[h_C^{MAP}(\epsilon)\right]$
   (b) Asymptotic BP EXIT: $h_{\mathcal{C}}^{BP}(\epsilon) \triangleq \lim_{l\to\infty} \lim_{n\to\infty} E_{\mathcal{C}_n}\left[h_{C,l}^{BP}(\epsilon)\right]$
      i. BP limit exists for standard "tree-like" ensembles
4. MAP/BP Thresholds
   (a) ϵ_{MAP} is max such that $h_{\mathcal{C}}^{MAP}(\epsilon) = 0$ for $0 < \epsilon < \epsilon_{MAP}$
   (b) ϵ_{BP} is max such that $h_{\mathcal{C}}^{BP}(\epsilon) = 0$ for $0 < \epsilon < \epsilon_{BP}$

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BP EXIT Function for (3,6) LDPC Ensemble

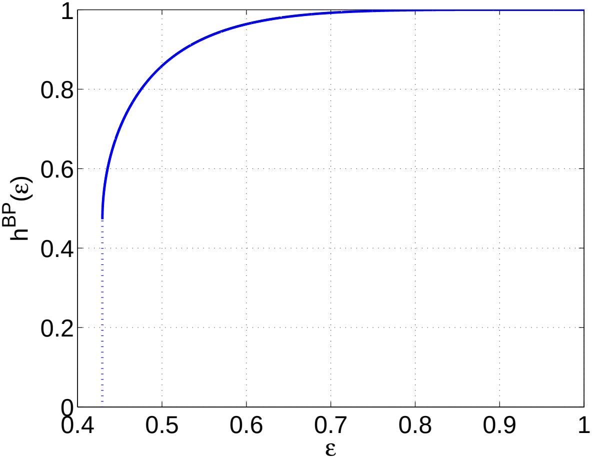

Figure: $(3,6)$-regular LDPC code on the erasure channel

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Upper Bounding the MAP Threshold

- For any code C defined by a parity-check matrix:

$$\begin{aligned} R_D &\leq R_C && \text{(design rate vs true rate)} \\ &= \int_0^1 h_C^{MAP}(\epsilon)\, d\epsilon && \text{(area theorem)} \\ &\leq \epsilon^* h_C^{MAP}(\epsilon^*) + \int_{\epsilon^*}^1 h_C^{MAP}(\epsilon)\, d\epsilon && \text{(non-decreasing)} \\ &\leq \underbrace{\epsilon^* h_C^{MAP}(\epsilon^*)}_{A} + \underbrace{\int_{\epsilon^*}^1 h_{C,l}^{BP}(\epsilon)\, d\epsilon}_{B} && \text{(MAP is optimal)} \end{aligned}$$

- For sequence $\mathcal{C}_n$, limsup of expectation for $\epsilon^* = \epsilon_{MAP} - \frac{1}{n}$ gives

$$R_D \leq \int_{\epsilon_{MAP}}^1 h_{\mathcal{C}}^{BP}(\epsilon) d\epsilon$$

- Proof: $E_{\mathcal{C}_n}[A] \to 0$ and $\limsup E_{\mathcal{C}_n}[B] \leq \int_{\epsilon_{MAP}}^1 h_{\mathcal{C}}^{BP}(\epsilon) d\epsilon$

Graphical Construction

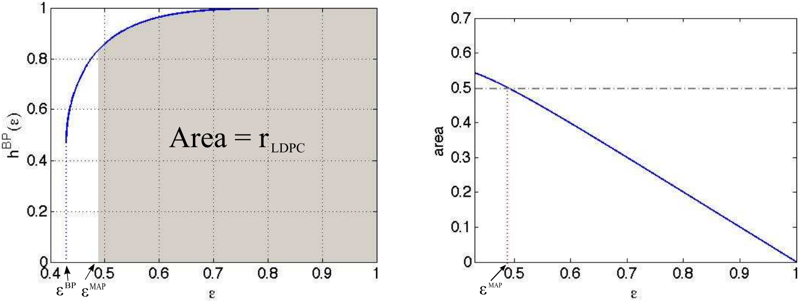

Figure: $(3,6)$-regular LDPC code on the erasure channel, $r_{LDPC} = 0.5$

$\epsilon^{BP} = 0.4294$; $\epsilon^{MAP} = 0.4881$ (MMU)

Lower Bounding the MAP Threshold

- For an LDPC code on the BEC
  - Decode with a peeling decoder until the decoder gets stuck
  - Residual graph is a function of channel erasure probability ϵ
    - Find ϵ^* where MAP decoding of residual code succeeds w.h.p.
    - Analysis based on residual-code weight-enumerator (WE)
  - This ϵ^* gives a lower bound on the MAP threshold
- Weight enumerator (WE) analysis of residual-graph code
  - Decoding succeeds if code has only all-zero codeword w.h.p.
  - Sufficiency can be numerically verified for each residual graph
  - MMU proved this analytically for regular LDPC codes

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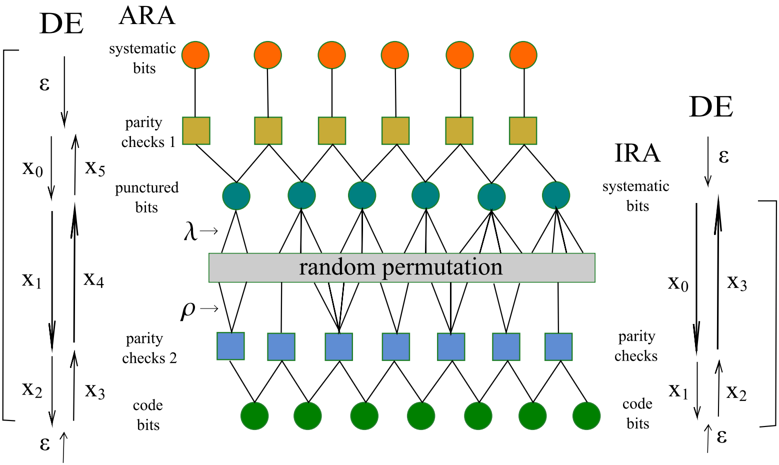
Gallager-Tanner-Wiberg Graph for ARA and IRA Codes



MAP Threshold Bounds for Systematic IRA (SIRA) Codes

- Density evolution for SIRA codes (Jin,Khandekar,McEliece)

$$\begin{aligned} x_0^{(l)} &= \epsilon\lambda(x_3^{(l)}) \\ x_1^{(l)} &= 1-(1-x_2^{(l-1)})R(1-x_0^{(l-1)}) \\ x_2^{(l)} &= \epsilon x_1^{(l)} \\ x_3^{(l)} &= 1-(1-x_2^{(l)})^2\rho(1-x_0^{(l-1)}) \end{aligned}$$

- *Design rate of the SIRA code ensemble*

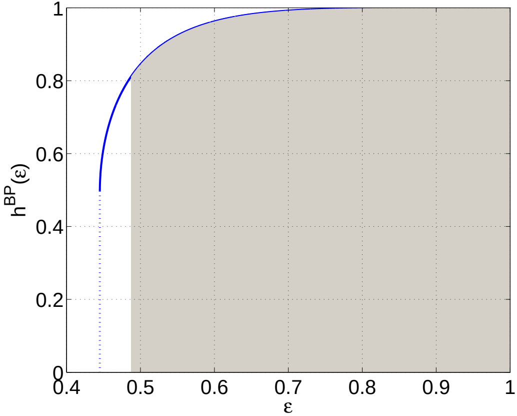
$$r_{IRA} = \left(1+\frac{\int_0^1 \rho\left(x\right)dx}{\int_0^1 \lambda\left(x\right)dx}\right)^{-1}$$

- *BP EXIT function* of the SIRA ensemble

$$h^{BP-IRA}\left(\epsilon\right) = r_{IRA}\,L\left(x_3\right) + \left(1-r_{IRA}\right)x_1^2$$

Upper Bound for (4,4) Regular SIRA Code

- DE/Integral for rate-$\frac{1}{2}$ SIRA ensemble: $(\lambda, \rho) = (x^3, x^3)$



$$\epsilon^{BP} = 0.4451;\ \epsilon^{MAP} \le 0.4872;\ \epsilon^{Shannon} = 0.5$$

Threshold Comparison for Various IRA Code Ensembles

$\lambda(x)$	$\rho(x)$	ϵ^{BP}	ϵ^{MAP}	$\epsilon^{Shannon}$	rate
x^2	x^2	0.4448	0.4651	0.5000	0.5000
x^3	x^3	0.4451	0.4872	0.5000	0.5000
x^4	x^4	0.4308	0.4946	0.5000	0.5000
x^2	x^5	0.2890	0.3078	0.3333	0.6667
x^3	x^7	0.2848	0.3249	0.3333	0.6667
x^4	x^9	0.2694	0.3301	0.3333	0.6667
x^2	x^8	0.2139	0.2303	0.2500	0.7500
x^3	x^{11}	0.2086	0.2438	0.2500	0.7500
x^4	x^{14}	0.1952	0.2477	0.2500	0.7500

Applying lower bound to each ensemble shows tightness

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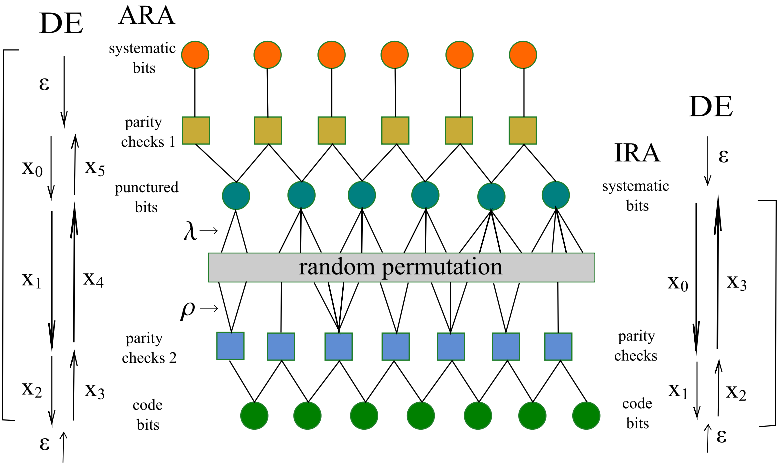
Gallager-Tanner-Wiberg Graph for ARA and IRA Codes



MAP Threshold Bounds for ARA Codes

- Density evolution for ARA code ensembles (Pfister and Sason)

$$\begin{aligned} x_0^{(l)} &= 1-\left(1-x_5^{(l-1)}\right)(1-\epsilon) & x_3^{(l)} &= \epsilon x_2^{(l)} \\ x_1^{(l)} &= \left(x_0^{(l)}\right)^2 \lambda\left(x_4^{(l-1)}\right) & x_4^{(l)} &= 1-\left(1-x_3^{(l)}\right)^2 \rho\left(1-x_1^{(l)}\right) \\ x_2^{(l)} &= 1-R\left(1-x_1^{(l)}\right)\left(1-x_3^{(l-1)}\right) & x_5^{(l)} &= x_0^{(l)} L\left(x_4^{(l)}\right) \end{aligned}$$

- Design rate of the ARA code ensemble

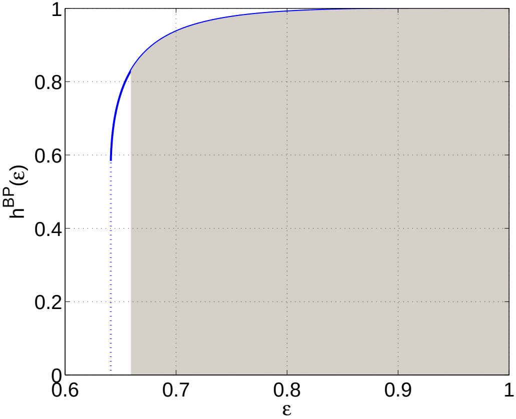
$$r_{ARA} = \frac{1}{1+\frac{L'(1)}{R'(1)}}$$

- *BP EXIT function* of the ARA code ensemble

$$h^{BP-ARA}(\epsilon) = r_{ARA}\left[1-(1-x_5)^2\right] + (1-r_{ARA})\, x_2^2$$

MAP Threshold Bounds for ARA Codes

- DE/Integral for rate-$\frac{1}{3}$ ARA ensemble: $(\lambda, \rho) = \left(x^2, \frac{2}{3}x + \frac{1}{3}\right)$



$$\epsilon^{BP} = 0.6412;\ \epsilon^{MAP} \leq 0.6593;\ \epsilon^{Shannon} = 0.6666$$

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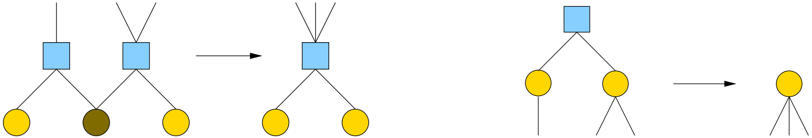
Graph Reduction for IRA/ARA Codes



- *Graph reduction* removes erasures from accumulator outputs
  - By merging the two check nodes adjacent to an erasure
  - Equivalent to summing check equations to remove erased bit
- *Graph reduction* removes degree-2 checks from accum. inputs
  - By absorbing equality constraint into a single bit node
  - Equivalent to removing an extraneous variable
- After graph reduction we have an LDPC code with a new d.d.
  - Resulting LDPC ensemble is standard given new d.d.
  - One can apply the standard peeling decoder analysis
- Introduced by Pfister and Sason

Lower Bound for IRA/ARA Codes on the BEC

- Graph reduction reduces the IRA/ARA code to an LDPC code
- Decode with a peeling decoder until the decoder gets stuck
    - Residual graph is a function of channel erasure probability ϵ
        - Find ϵ^* where MAP decoding of residual code succeeds w.h.p.
        - Analysis based on residual-code weight-enumerator (WE)
    - This ϵ^* gives a lower bound on the MAP threshold
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System Description

Figure: Block Diagram of the System

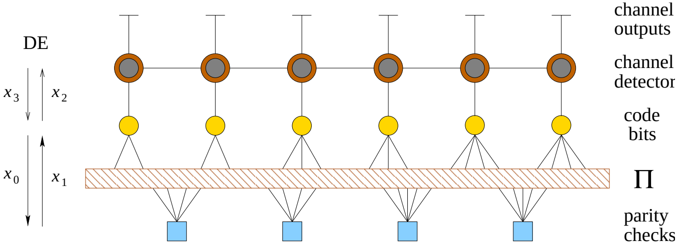
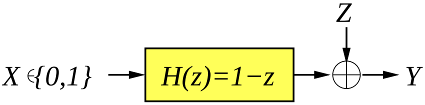
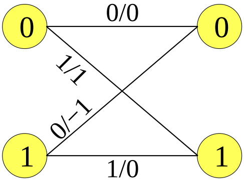


Figure: Gallager-Tanner-Wiberg graph of the joint iterative decoder

Channel Model and Erasure Noise

- The Dicode Erasure Channel (DEC) Model
  - Discrete-time channel with linear response $H(z) = 1 - z$
  - Output erased independently with probability ϵ
  - State given by last input, edges labelled by input/output





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Definition (MAP Joint Decoding EXIT Function)

Let C be a length-n binary code defined by $p_{X_1^n}(x_1^n)$. Codeword X_1^n is chosen according to $p_{X_1^n}(x_1^n)$, Z_1^n is the noiseless output of the ISI channel with initial state S_1, and Y_1^n is the result of transmitting Z_1^n over a BEC(ϵ). Then, the *MAP-JD EXIT function* is

$$h_C^{MAP-JD}(\epsilon) \triangleq \frac{1}{n}\sum_{i=1}^{n} H\left(Z_i | Y_1^n(\epsilon) \setminus Y_i(\epsilon), S_1\right).$$

Theorem (Area Theorem)

$$\frac{1}{n} H\left(X_1^n | Y_1^n(\epsilon), S_1\right) = \int_0^{\epsilon} h_C^{MAP-JD}(\delta)\, d\delta$$

BP EXIT Function for Joint Decoding

- Density evolution for joint decoding (Pfister and Siegel)

$$\begin{aligned} x_0^{(l+1)} &= x_3^{(l)}\lambda(x_1^{(l)}) & x_2^{(l+1)} &= L(x_1^{(l+1)}) \\ x_1^{(l+1)} &= 1-\rho(1-x_0^{(l+1)}) & x_3^{(l+1)} &= f(x_2^{(l+1)};\epsilon) \end{aligned}$$

- BP-JD EXIT function from analysis of BCJR for channel
- Asymptotic BP-JD EXIT function for DEC(ϵ)

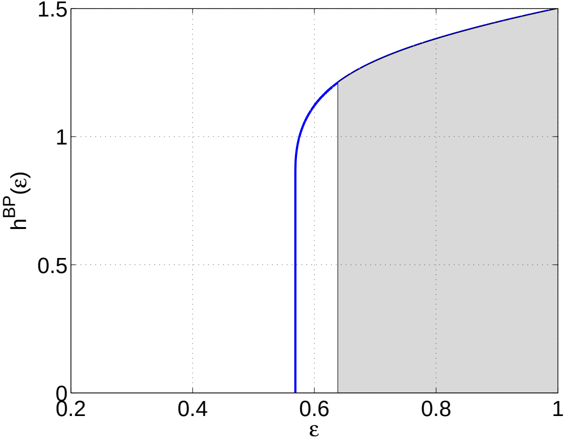
$$h^{BP-JD}(\epsilon) = \frac{2\epsilon x_3 \left(4+\epsilon x_3 - 2x_3\right)}{\left(2-x_3\left(1-\epsilon\right)\right)^2}$$

  - where $x_3 \triangleq \lim_{l\to\infty} x_3^{(l)}$

Upper Bound on the MAP Threshold

(3,6)-regular LDPC code with rate $= \frac{1}{2}$

Joint Decoder for Dicode Erasure Channel



$$\epsilon^{BP} = 0.5689;\ \epsilon^{MAP} \le 0.6386$$

Summary

- Analyzed the MAP threshold of 3 iterative decoding systems
  - MAP Threshold upper/lower bounds for IRA/ARA Codes
  - MAP Threshold upper bound for joint decoding of LDPC/ISI
- Tight LB implies $h_{\mathcal{C}}^{MAP}(\epsilon) = h_{\mathcal{C}}^{BP}(\epsilon)$ a.e. for $\epsilon > \epsilon_{MAP}$
- Results quantify iterative decoding loss for IRA/ARA codes

Future Work

- MAP Threshold lower bounds (i.e., tightness of the UB)
    - Prove tightness analytically for bit-regular IRA/ARA codes
    - Define peeling decoder for LDPC/ISI joint decoder to prove LB
- Extensions to non-erasure channels
    - Upper bound for IRA/ARA follows from GEXIT functions
    - Upper bound for joint decoder possible but requires work
    - Lower bound is still an open problem for LDPC codes