Information Theory and Coding for Compressed Sensing

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Outline

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What is Compressed Sensing? (1)

- Compressed sensing (CS) is a relatively new area of signal processing and statistics that focuses on signal reconstruction from a small number of linear projection measurements
- CS originated with the observation that many systems:
  - (1) Sample a large amount of data
  - (2) Perform a linear transform (e.g., DCT, wavelet, etc...)
  - (3) Throw away all the small coefficients
- If the locations of the important transform coefficients were known in advance, then they could be sampled directly with reduced complexity

What is Compressed Sensing? (2)

- Compressed sensing (CS) emerged when it was observed that random sampling could be used (with a small penalty) to achieve the sample rate reduction result without knowing the locations of important coefficients
- CS has two stages: sampling and reconstruction
  - During sampling, the signal-of-interest (SOI) is sampled by computing its projection onto a set of sampling vectors
  - During reconstruction, the SOI is estimated from the samples
  - In many cases, the number of samples required for a good estimate is much smaller than other methods (e.g., Nyquist sampling at twice the maximum frequency)

Basic Problem Statement

- Use m dot-product samples to reconstruct a signal
  - The signal vector is $x \in \mathbb{R}^n$
  - The $m \times n$ measurement matrix is $\Phi \in \mathbb{R}^{m \times n}$
  - The length-m sample vector is $y = \Phi x$

Basic Problem Statement

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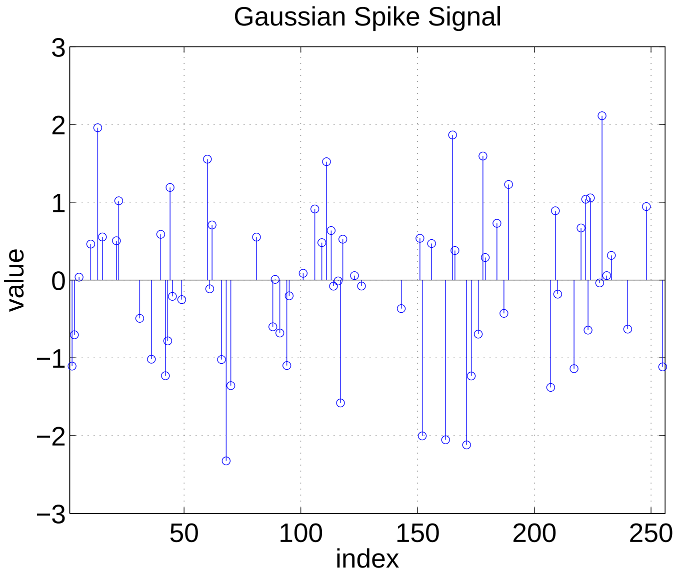
  - The length-m sample vector is $y = \Phi x$
- Given y, the valid signal set is $V(y) = \{x' \in \mathbb{R}^n | \Phi x' = y\}$
  - If $m < n$, then a unique solution is not possible
  - With prior knowledge, we try to choose a "good" solutions
  - If x is sparse (w.r.t. Hamming weight $\|\cdot\|_H$), then

$$\hat{x} = \arg \min_{x' \in V(y)} \|x'\|_H$$

Exactly Sparse Signal



Reconstruction

- "Basis Pursuit" by Chen, Donoho, and Saunders (1998)
  - Constrained $\|\cdot\|_H$ minimization is NP-Hard
  - Instead, use LP to solve the relaxation: $\min_{x' \in V(y)} \|x'\|_1$

- The minimizing vector is the same (w.h.p.) for both problems!
  - If Φ is chosen randomly (e.g., such that $\Phi_{ij} \sim N(0,1)$)
  - And m is large enough: $m \geq C \|x\|_H \log n$
    - ★ Or equivalently, x is very sparse: $\|x\|_H \leq \frac{m}{C \log n}$
  - See work by Donoho, Candès, Romberg, Tao, and others

Drawbacks of Basis Pursuit

- Reconstruction Performance
  - The set $\{x \in \mathbb{R}^n | \, \|x\|_1 \leq A\}$ is bigger than $\left\{x \in \mathbb{R}^n | \, \|x\|_{0^+} \leq A\right\}$
  - Many more measurements required to reconstruct the signal
  - Solution: Instead minimize $\|x\|_p$ for $0 < p < 1$
    - Reweighted ℓ_p algorithms and coding-based techniques

- Reconstruction Complexity
  - LP (even with fast transforms) takes $\Omega\left(n^2\right)$ operations
  - Solution: Fast approximate ℓ_1 algorithms
    - Bregman iteration and fixed point continuation
    - Variations of matching pursuit

Outline

Compressed Sensing and Coding

Compressed Sensing

- Signal: $x \in \mathbb{R}^n$
  - sparse: $\|x\|_H \leq \delta n$
- Measurement matrix:
  $$\Phi \in \mathbb{R}^{m \times n}$$
  - Blind to nullspace of Φ
- Sample vector: $y = \Phi x$
- Dec: $\hat{x} = \arg \min_{x' : y = \Phi x'} \|x'\|_H$

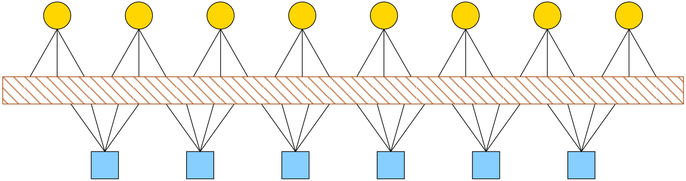
Linear Codes

- Error pattern: $e \in F^n$
  - sparse: $\Pr(e_i \neq 0) = \delta$
- Parity-check matrix:
  $$H \in F^{m \times n}$$
  - Code is nullspace of H
- Syndrome: $s = He$
- Dec: $\hat{e} = \arg \min_{e' : s = He'} \|e'\|_H$

Related Problems

- Sparse Approximation
  - Given a matrix Φ, find a sparse x such that $\|y - \Phi x\|$ is small
  - Problem studied in signal processing for quite some time
  - It is the problem that motivated basis pursuit (BP)
- Compressed Sensing
  - Choose a matrix Φ and decoder $D(y)$ such that:
    - ⋆ $D(\Phi x) \approx x$ when x is sufficently sparse
- From a coding perspective, the difference is who picks the code
  - Spase approximation: code is given
  - Compressed sensing: code can be optimized
  - Recent advances in coding favor the CS problem

Low-Density Parity-Check (LDPC) Codes



- Linear codes with sparse parity-check matrix $H \in F^{m \times n}$
  - Codewords defined to be $x \in F^n$ such that $\sum_j H_{ij} x_j = 0$
- Bipartite graph representation
  - An edge connects equation node i to symbol node j if $H_{ij} \neq 0$
  - Iterative decoding passes messages along edges of graph
  - Graphs with irregular degree profiles can approach capacity

Coding Over Large Finite Alphabets

- Verification-based iterative decoding of LDPC codes
  - Introduced by Luby and Mitzenmacher (Allerton02)
  - A message is "verified" if correct with high probability
    - Message has value $Z \in \mathrm{GF}(q)$ and status V or U

- For the q-ary symmetric channel with large q: $(C \approx 1 - \delta)$

$$\Pr(y|x) = \begin{cases} 1-\delta & \text{if } x = y \\ \delta/(q-1) & \text{if } x \neq y \end{cases} \quad x, y \in \mathrm{GF}(q)$$

  - Verify msg if two independent observations match
  - Pr(two ind. observations give same incorrect value) = $\frac{1}{q-1}\delta^2$
  - Verify if all symbols nodes in a parity check sum to zero
  - False Verification (FV): message is *verified* but not correct

Outline

Verification-Based Decoding Algorithms

- Algorithms by Luby and Mitzenmacher (Allerton02,IT05)
  - 1st Algorithm (denoted LM1)
    - ★ Verify all symbols whose parity check sums to zero
    - ★ Use degree-1 equations to determine symbol values
  - 2nd Algorithm (denoted LM2)
    - ★ Verify all symbols whose parity check sums to zero
    - ★ Use degree-1 equations to determine symbol values
    - ★ Verify messages if any two match at symbol node
- Connection between CS and Codes over Real Numbers
  - Basic assumption: "verified symbols are correct w.h.p."
  - Holds both for large finite-fields and the real numbers

CS Reconstruction via LM1

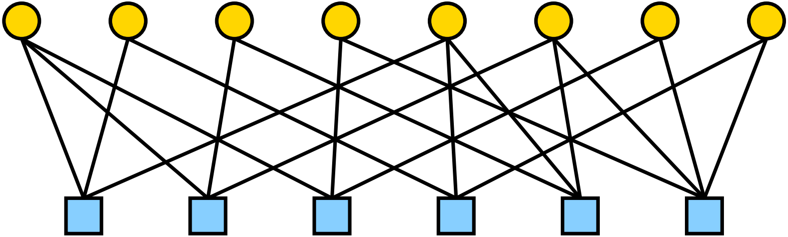
Example

Let Φ be the following parity-check matrix H

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

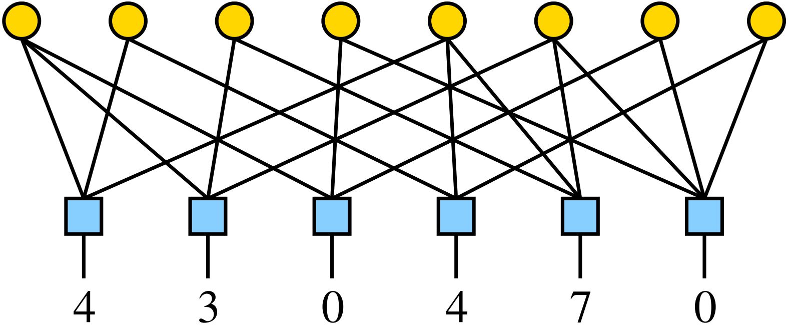
Note: We choose edge weights of one for simplicity

CS Reconstruction via LM1



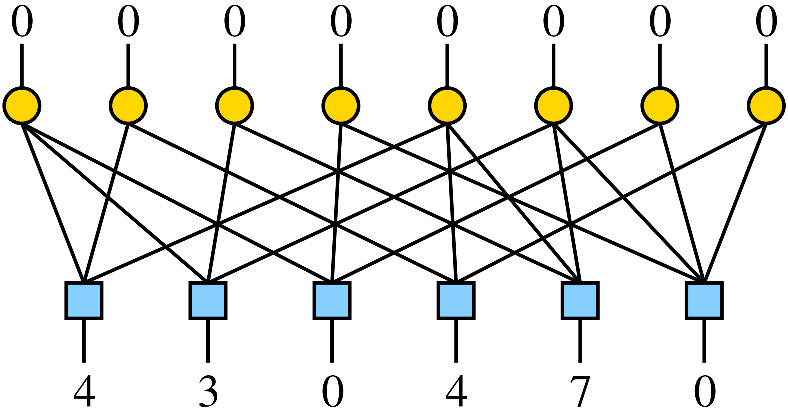
Signal (circles) measurement (squares) model

CS Reconstruction via LM1



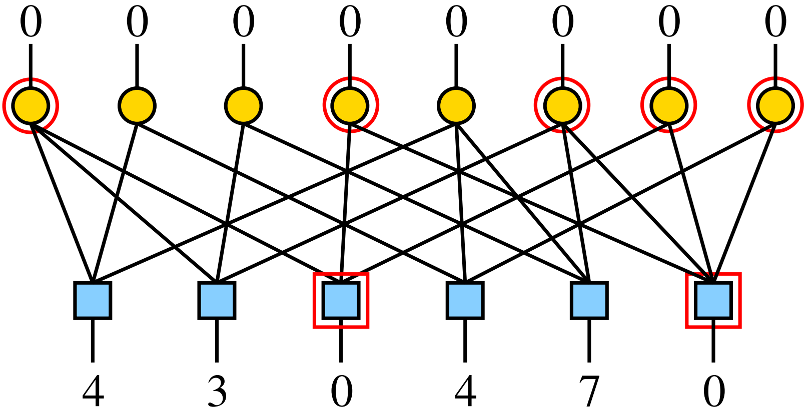
With measurements exposed

CS Reconstruction via LM1



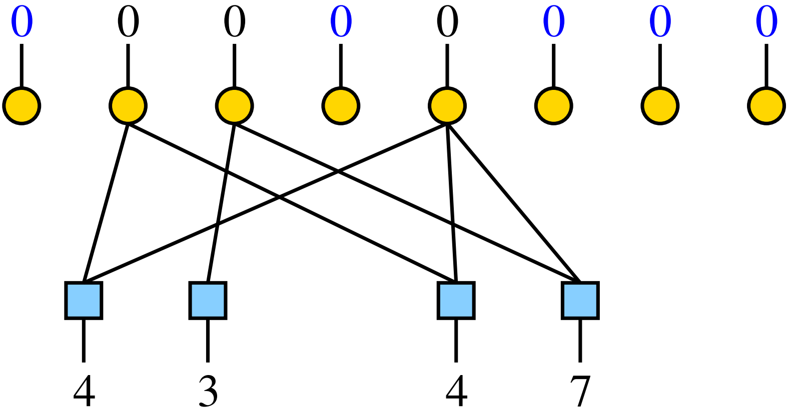
All-zero signal assumed for decoding

CS Reconstruction via LM1



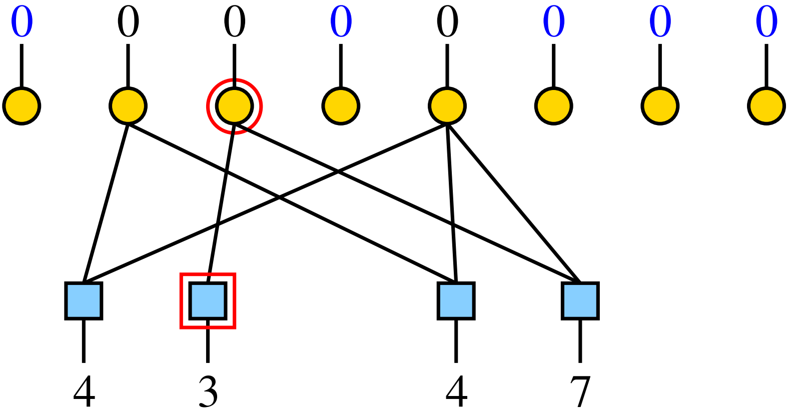
Assume all satisfied checks are correct

CS Reconstruction via LM1



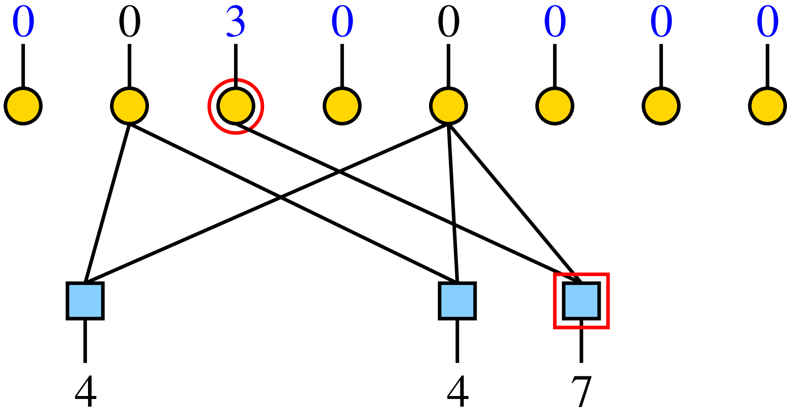
Remove edges and fix values

CS Reconstruction via LM1



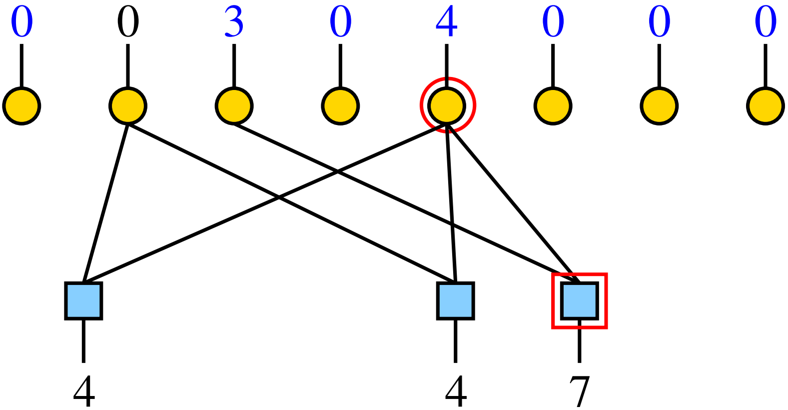
Use degree-1 check to determine a variable

CS Reconstruction via LM1



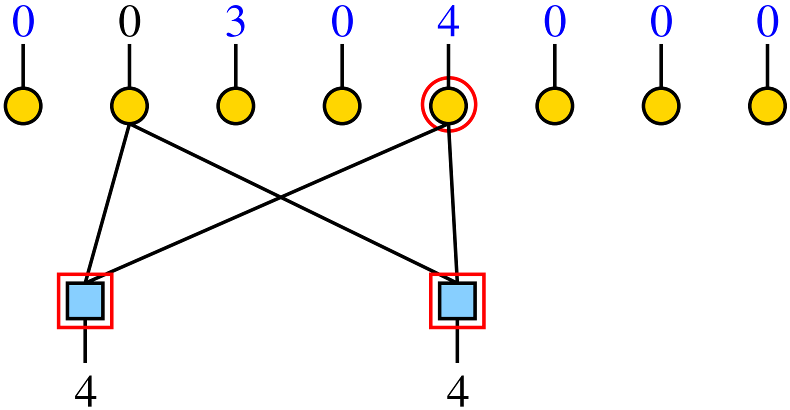
This value can be removed from all equations

CS Reconstruction via LM1



Another variable is determined

CS Reconstruction via LM1



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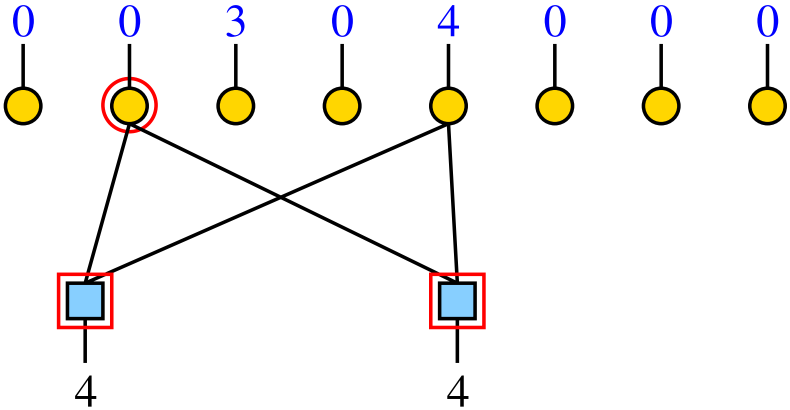
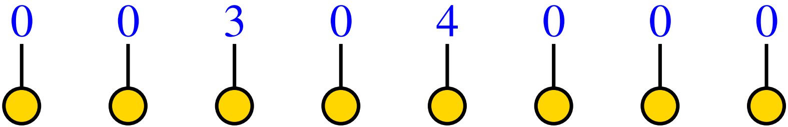
CS Reconstruction via LM1



Final value is determined in two ways

CS Reconstruction via LM1



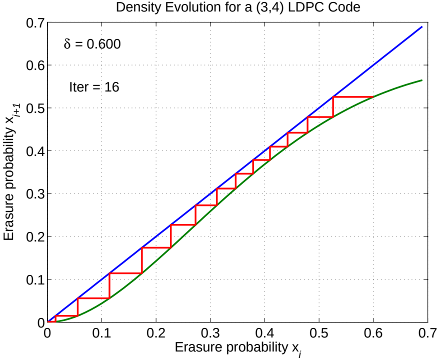
Reconstruction is successful

Sudocodes and Belief Propagation Approaches

- "Sudocodes" of Sarvotham, Baron, and Baraniuk (ISIT06)
  - Measurement matrix Φ is a sparse 0/1 matrix (L ones per row)
  - Iterative decoding method based on matching coefficients
    - ★ Equivalent to LM2 for CS problems
  - Requires $m = O(n \log n)$ for coverage (e.g., balls/bins)
  - Second phase used to handle uncovered elements
- LDPC/Belief propagation for CS by Sarvotham et al. (2006)
  - Sparse measurement matrix has -1/0/+1 entries
  - Approx. sparse signal modeled by mixture of two Gaussians
  - Belief propagation decoding used to find important coefficients
  - Algorithm evaluated primarily by simulation
- Expander LDPC codes for CS by Xu and Hassibi (ITW07)

Outline

Warmup Example: The Binary Erasure Channel



Density evolution for a (j,k)-regular LDPC code

$$x_{i+1} = \delta \left(1 - (1 - x_i)^{k-1}\right)^{j-1}$$

Code	Rate	δ^*
(3,6)	0.5	0.4294
(3,12)	0.75	0.2105
(3,24)	0.875	0.1038

- Binary erasure channel with erasure prob. δ
- DE tracks the message erasure rate x_i after i iterations
- Gives noise threshold δ^* below which decoding succeeds w.h.p.
- Requires numerical calculations for each code ensemble

Very Sparse <=> Very High Rate

- CS often considers the sublinear sparsity $\delta n = o(n)$ regime
  - How does the DE threshold scale in this regime?
  - Shannon limit says that $\delta^* \leq j/k$
  - To understand this, numerical DE is not enough

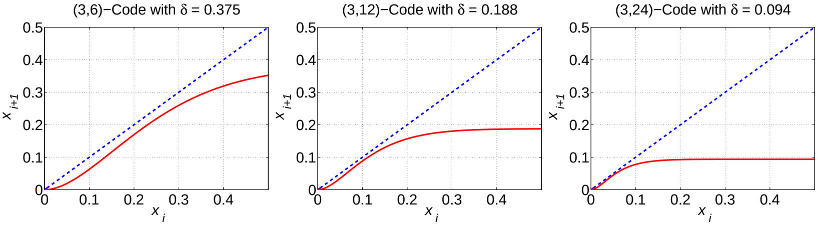
Very Sparse <=> Very High Rate

- CS often considers the sublinear sparsity $\delta n = o(n)$ regime
  - How does the DE threshold scale in this regime?
  - Shannon limit says that $\delta^* \leq j/k$
  - To understand this, numerical DE is not enough
- Let $\alpha = \dfrac{j\delta}{k-1}$ and $x_i = \dfrac{jy_i}{k-1}$ so that

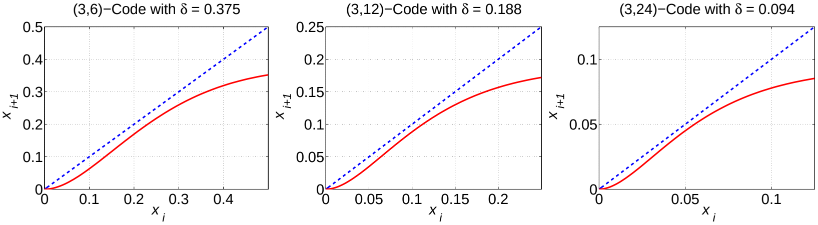
$$\begin{aligned} y_{i+1} &= \frac{j}{k-1}\delta\left(1-(1-x_i)^{k-1}\right)^{j-1} \\ &= \alpha\left(1-\left(1-\frac{jy_i}{k-1}\right)^{k-1}\right)^{j-1} \\ &\searrow \alpha\left(1-e^{-jy_i}\right)^{j-1} \end{aligned}$$

- Expression independent of k

DE Scaling Illustrated



Unscaled



Scaled

DE Scaling Analysis for the BEC

Theorem: DE Scaling for the BEC

For $\theta < 1$ and $j \geq 3$, there are constants $\overline{\alpha}_j$ and $k_0(j,\theta)$ such that

- for all $k \geq k_0(j,\theta)$ and $\delta \leq \theta \overline{\alpha}_j \left(j/(k-1)\right)$,
- iterative decoding on a BEC(δ) succeeds (w.h.p. as $n \to \infty$)
- using a randomly chosen (j,k)-regular LDPC code

- Example: $\overline{\alpha}_3 = 0.8184$ and $\theta = \frac{0.75}{0.8184}$ gives $k_0 = 9$
  - Implies $(3,k)$ codes achieve 75% of Shannon limit for all $k \geq 9$

- Proof follows from scaled DE and the concentration theorem

Reconstruction Guarantees for CS and DE

- DE for LM1: x_i = fraction of incorrect messages iteration i

$$x_{i+1} = \delta \left(1 - \left[1 - (1-\delta)\left(1-(1-x_i)^{k-1}\right)^{j-1} - x_i\right]^{k-1}\right)^{j-1}$$

  - Bad news: linear scaling $x_i \sim \frac{y_i}{k}$ fails!
  - Needs $x_i = \frac{\overline{\alpha}_j y_i}{(k-1)^{j/(j-1)}}$ and find that $\delta \sim \frac{\overline{\alpha}_j}{(k-1)^{j/(j-1)}}$

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- DE Reconstruction Guarantee: "Randomized Reconstruction"
  - Signal and code chosen randomly $\implies$ Success w.h.p. as $n \to \infty$

- BP Reconstruction Guarantee: "Uniform Reconstruction"
  - There exists Φ that works for all signals $\|x\|_H \leq \frac{m}{C \log n}$

DE Scaling Analysis of LM1 Decoding for CS

- Probabilistic signal model
  - Length-n vector x with i.i.d. elements: $\Pr(x_i \neq 0) = \delta$
  - $\|x\|_H \leq (1+\epsilon)\delta n$ with high probability as $n \to \infty$
- Sparse measurement matrix with m rows
  - Regular degree profile with j non-zero entries per column
  - Non-zero entries drawn from a continuous distribution

Theorem: Randomized Reconstruction with LM1

CS with (j,k)-regular LDPC codes and LM1 decoding provides randomized reconstruction (w.h.p. as $n \to \infty$) if $\delta \leq (k-1)^{-j/(j-1)}$.

- Oversampling ratio $\leq e \left\lceil \ln \frac{1}{\delta} \right\rceil$ with linear complexity in n
  - If symbol node degree $j = \left\lceil \ln \frac{1}{\delta} \right\rceil$ increases as $\delta \to 0$

Is Linear Scaling Possible for CS?

- "Linear Scaling" = $C\|x\|_H$ measurements suffice for all $\|x\|_H$
  - Yes: Exhaustive search, random codes, and strictly sparse
  - Not possible with noise or approximately sparse signals
- Information-theoretic analysis: $(j, jn^{1-\omega})$-regular codes
  - Random signal X_1^n with measurements Y_1^m where $m = n^\omega$
    - ⋆ Non-zero entries of X_1^n drawn i.i.d. ~ Z
    - ⋆ Average number of non-zero entries per "check" is λ
  - Reconstruction requires $H(Y_1^m) \geq H(X_1^n)$ which holds only if

$$H(Z) \geq \frac{\omega}{j-1} \log \frac{n}{\lambda}$$

    - ⋆ Linear scaling essentially requires $H(Z) = \infty$ (e.g., not discrete)

LM2 Achieves Linear Scaling

- Probabilistic signal model
  - Length-n vector x with i.i.d. elements: $\Pr(x_i \neq 0) = \delta$
  - $\|x\|_H \leq (1+\epsilon)\delta n$ with high probability as $n \to \infty$
- Sparse measurement matrix with m rows
  - Regular degree profile with $j = 4$ non-zero entries per column
  - Non-zero entries drawn from a continuous distribution

Theorem: Randomized Reconstruction with LM2

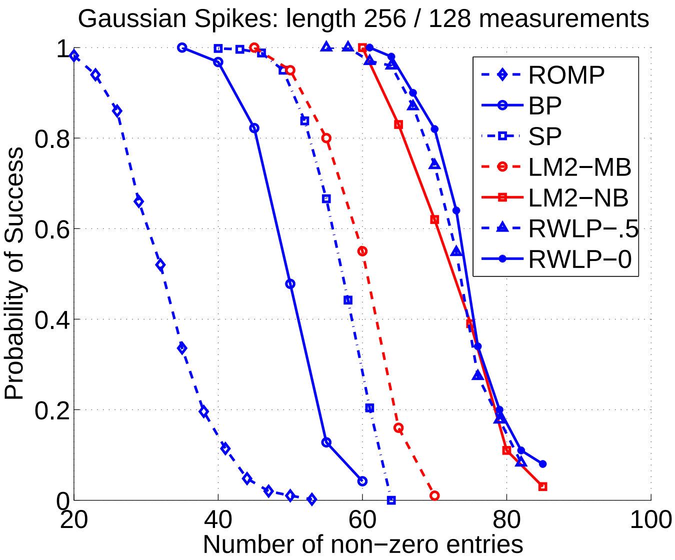
CS with $(4, k)$-regular LDPC codes and LM2 decoding (for $k \geq k_0$) provides randomized reconstruction (w.h.p. as $n \to \infty$) if $\delta \leq \frac{1}{3}(4/k)$

- Constant oversampling ratio of 3 with linear complexity in n

Comparison With Other Algorithms

- Regularized Orthgonal Matching Pursuit (ROMP)
  - Needell and Vershynin (2007)
- Basis Pursuit (BP)
  - Chen, Donoho, and Saunders (1998)
- Subspace Pursuit (SP)
  - Dai and Milenkovic (2008)
- Message-passing verification decoding (LM2-MB)
- Peeling-style verification decoding (LM2-NB)
  - Zhang and Pfister (2008)
- Iteratively Reweighted ℓ_p (RWLP-p)
  - Chartrand and Yin (2008)
- Gaussian Φ used except LM2-MB/LM2-NB use sparse $(3, 6)$

Simulation Results



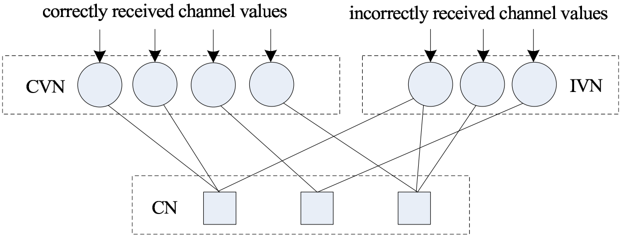
(Spikes have random locations and standard Gaussian amplitudes)

Outline

Stopping Set Analysis for the BEC

- A stopping set (s.s.) for iterative decoding on the BEC is
  - "a set of bit nodes connected even # of times to check nodes"
  - Iterative decoding progresses until it reaches a stopping set
- Uniform erasure correction
  - Let s_{min} be the size of the smallest stopping set
    - ⋆ Iterative decoding corrects all erasure patterns weight < s_{min}
  - For $j \geq 3$, random LDPC codes have $s_{min} = \gamma n$ w.h.p.
- One can also derive a scaling law for stopping sets
  - (j, k)-codes with large k have $s_{min} \sim ne(k-1)^{-j/(j-2)}$ w.h.p.

Stopping Set Analysis for LM1



- Peeling decoder removes symbols as they are verified
  - A check is removed if all edges are verified or correct
  - A symbol is verified if it is attached to a degree-1 check
- If neither condition occurs, the configuration is a stopping set

Uniform-in-Probability Reconstruction with LM1

- Signal Model
  - All length-n vectors x such that $\|x\|_H \leq \delta n$
- Sparse measurement matrix with m rows
  - (j,k)-regular code with fixed connections
  - Non-zero entries drawn from a continuous distribution
    - ★ Random for each x implies no false verification (FV) w.h.p.

Theorem: Uniform-in-Probability Reconstruction with LM1

There exists a (j,k)-regular LDPC code with $k > k_0$ such that LM1 succeeds for all x s.t. $\|x\|_H < nC_j(k-1)^{-j/(j-2)}$ assuming no FV.

- Based on combinatorial analysis of LM1 stopping sets

Summary

- Discussed connections between ECC and CS
  - Some basic connections now clear
  - Density evolution allows threshold analysis
  - Stopping-set analysis allows stronger guarantees
- Linear-time recovery algorithms for CS based on coding
  - First approach with a constant oversampling ratio

Open Problems

- Noisy observations and approximately sparse signals
  - Can one extend both the signal model and analysis?
- Signal is sparse is some basis unknown at encoder
  - Encoder cannot tranform into sparse basis before sampling
  - Can one add transform to decoding graph (e.g., ISI channels)?
- Code design for specific applications
  - Can protograph and other code designs be used for CS?
- Can one extend the DE scaling law to general channels?