Reed-Muller Codes Achieve Capacity on Erasure Channels

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Reed-Muller Codes (I)

- Codes by Muller, adecoder by Reed, both in 1954
  - Multivariate polynomial-evaluation codes over binary field
  - Minimum distance $\approx \sqrt{N}$ (Not so good!)
- Very popular in theoretical computer science (TCS)
  - locally decodable, locally testable, probabilistic proof systems
- Capacity-Achieving Conjectures
  - By Shu Lin: "RM Codes are Not So Bad" (Tokyo ITW, 1988)
  - By Costello and Forney for Rate-$1/2$ and BI-AWGN, 2007
- First known conjecture in print by Dumer and Farrell in 1994
  - They show BCH codes achieve capacity on BEC as rate $\to 1$
  - Open problem stated for Reed-Muller codes at constant rates

Reed-Muller Codes (II)

- Closely related to polar codes
  - From Hadamard matrix, one choice of rows generates Reed-Muller and some other polar codes

In fact Arikan remarked:

> *It is interesting that the possibility of RM codes being capacity-achieving codes under ML decoding seems to have received no attention in the literature*

- Under MAP, Reed-Muller observed to be better than polar (Arikan and Mondelli-Hassani-Urbanke)

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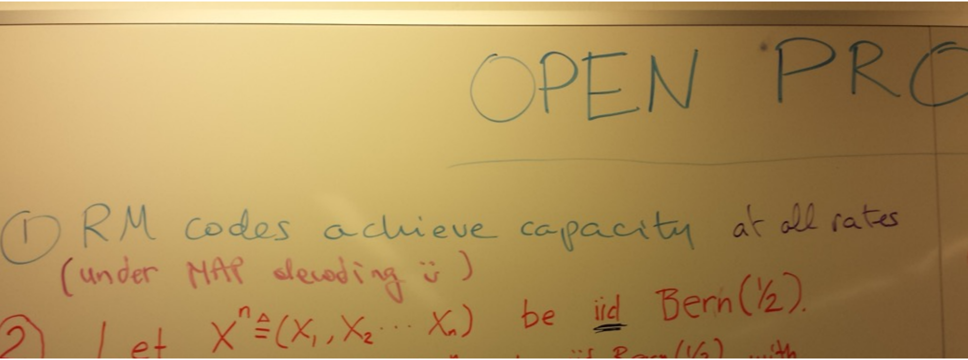
- In 2014, Abbe-Shpilka-Wigderson showed capacity achieving for rates $\to 0, 1$ (erasures) and rates $\to 0$ (errors)
- Can they achieve capacity for constant rates?


OPEN PRO
① RM codes achieve capacity at all rates
(under MAP decoding)
② Let $X^n \triangleq (X_1, X_2 \cdots X_n)$ be iid Bern(½).

Can They?

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YES!

Let $\{\mathcal{C}_n\}$ be a sequence of codes with rates $r_n \to r \in (0, 1)$

- Blocklengths $N_n \to \infty$
- Suppose the permutation group of $\mathcal{C}_n$ is doubly transitive and
- Then, $\{C_n\}$ achieves capacity on the BEC under bit-MAP

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Important Consequences

- Reed-Muller codes achieve capacity
- Primitive narrow-sense BCH codes achieve capacity
- Affine-invariant codes achieve capacity
- Extends to block-MAP for Reed-Muller and BCH
- By Kumar-Pfister and Kudekar-Mondelli-Sasoglu-Urbanke

Few Remarks

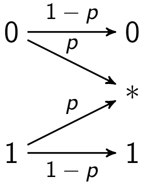
- Scope of the work
  - Linear Codes, Erasure Channels, MAP Decoding
- Buzzwords
  - EXIT functions, monotone boolean functions, k-transitivity
- Amalgamation
  - EXIT functions (from iterative decoding)
  - Automoprhism/Permutation groups (from algebraic coding)
  - Monotone boolean functions (from computer science)
- Why do they achieve capacity?

Proof

Basic Setup

- Binary linear code $\mathcal{C} \subset \{0,1\}^N$ is a K-dim. subspace of $\mathbb{F}_2^N$
- Binary Erasure Channel, parametrized by p

$\mathcal{X} = \{0,1\}$



$\mathcal{Y} = \{0,1,*\}$

- $\underline{X} = (X_1, \ldots, X_N) \longleftrightarrow$ uniform codeword from $\mathcal{C}$
- $\underline{Y} = (Y_1, \ldots, Y_N) \longleftrightarrow$ received sequence from $\underline{X}$

MAP Decoding on Erasure Channels

Set of Consistent Codewords

$$\mathcal{C}(\underline{y}) = \{\underline{x} \in \mathcal{C} \quad | \quad x_i = y_i \quad \text{when } y_i \neq *\}$$

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- $|\mathcal{C}(\underline{y})| = 1 \longleftrightarrow$ one can recover codeword $\underline{X}$

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- Otherwise
  - Half of codewords in $\mathcal{C}(\underline{y})$ have $x_i = 0$ and half have $x_i = 1$
  - uniform codeword $\longleftrightarrow$ uniform posterior
  - $H(X_i|\underline{Y} = \underline{y})) = 1$

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  - $H(X_i|\underline{Y} = \underline{y})) = 1$
- $H(X_i|\underline{Y} = \underline{y})$ is either 0 or 1 (Boolean)

MAP Decoding on Erasure Channels: Prob. of Bit-Error

$$\text{Msg.} \longrightarrow \boxed{\mathcal{C}} \longrightarrow \underline{X} \longrightarrow \boxed{\mathrm{BEC}(p)} \longrightarrow \underline{Y}$$

Error Prob. of bit X_i

- Bit-MAP decoder $D_i : \mathcal{Y}^N \to \mathcal{X} \cup \{*\}$
- Error prob. of bit i: $P_{b,i} = \Pr(D_i(\underline{Y}) = *)$
- Average bit error prob. $P_b = \frac{1}{N} \sum_i P_{b,i}$

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Error Prob. as Entropy

$$H(X_i|\underline{Y}) = \sum_{\underline{y}} \Pr(\underline{Y} = \underline{y}) H(X_i|\underline{Y} = \underline{y}) = P_{b,i}$$

$$P_{b,i}(p) = H(X_i|\underline{Y}) \qquad\qquad P_b(p) = \frac{1}{N} \sum_i H(X_i|\underline{Y})$$

Implicit parametrization by channel erasure probability p

Capacity-Achieving Codes on Erasure Channels

Suppose $\{\mathcal{C}_n\}$ is a sequence of codes with rates $r_n \to r \in (0, 1)$

If $P_b^{(n)}(p) \to 0$ for all $p < 1 - r$,

then $\{\mathcal{C}_n\}$ is Capacity-Achieving

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Remarks

- $\mathcal{C}$ has length N, K info bits, and $N - K$ parity bits
- Rate $r = K/N$ and redundancy $1 - r = (N - K)/N$
- Must correct almost all patterns with fraction $1 - r$ erasures
- Strong Requirement!

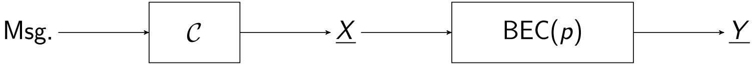
MAP EXIT Functions

Msg. $\longrightarrow \boxed{\mathcal{C}} \longrightarrow \underline{X} \longrightarrow \boxed{\text{BEC}(p)} \longrightarrow \underline{Y}$

EXtrinsic Information Transfer Function

- A popular tool in the iterative decoding community
- In 1999, introduced by ten Brink to visualize iterative decoding
- In 2003, formalized by Ashikhmin, Kramer, ten Brink

MAP EXIT Functions



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Definition

(Bit-i EXIT Function) $$h_i(p) = H(X_i \mid \underline{Y}_{\sim i})$$

(Average EXIT Function) $$h(p) = \frac{1}{N} \sum_i h_i(p)$$

- $\underline{Y}_{\sim i} = (Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_N)$
- Parameterized by channel erasure probability p

EXIT Functions: Bit-Erasure Probability

$$h_i(p) = H(X_i \mid \underline{Y}_{\sim i})$$

$$P_{b,i} = H(X_i \mid \underline{Y})$$

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$$\begin{aligned} H(X_i|\underline{Y}) &= \Pr(Y_i = *)H(X_i|\underline{Y}_{\sim i}, Y_i = *) \\ &\qquad + \Pr(Y_i = X_i)H(X_i|\underline{Y}_{\sim i}, Y_i = X_i) \\ &= pH(X_i|\underline{Y}_{\sim i}) \end{aligned}$$

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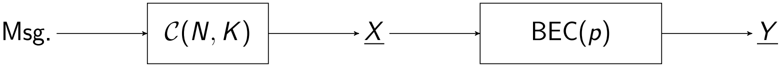
$$P_{b,i}(p) = ph_i(p) \qquad\qquad P_b(p) = ph(p)$$

EXIT Functions: Area Theorem



$$h_i(p) = H(X_i | \underline{Y}_{\sim i}) \qquad h(p) = \frac{1}{N} \sum_i h_i(p)$$

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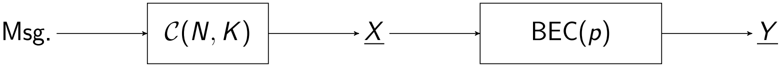

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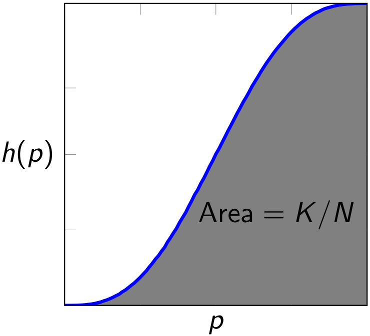
Area Theorem

$$\int_0^1 h(p)dp = K/N$$

- Conservation Principle
- Not satisfied by P_b

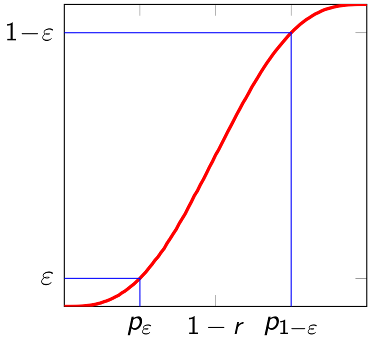


Capacity and EXIT Functions

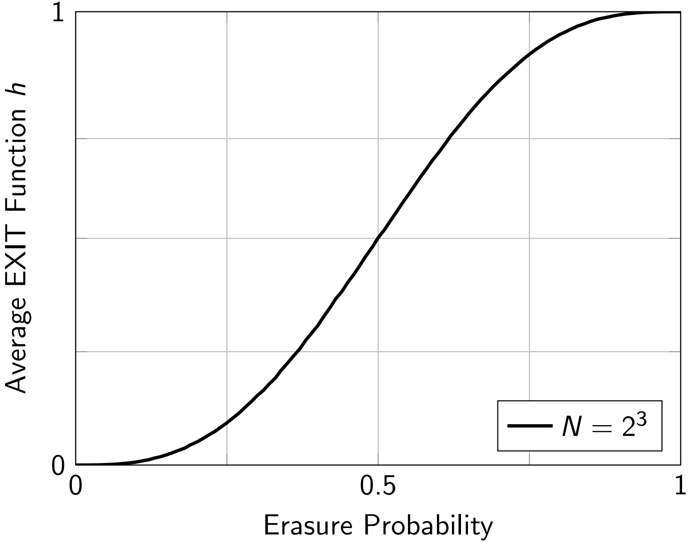
Suppose $\{\mathcal{C}_n\}$ is a sequence of codes with rates $r_n \to r$

The following are equivalent

- $\{\mathcal{C}_n\}$ achieves capacity
- $h^{(n)} \to \begin{cases} 0, & \text{if } p < 1-r, \\ 1, & \text{if } p > 1-r. \end{cases}$
- For all $\varepsilon > 0$, $p^{(n)}_{1-\varepsilon} - p^{(n)}_{\varepsilon} \to 0$



Rate-1/2 Reed-Muller Codes



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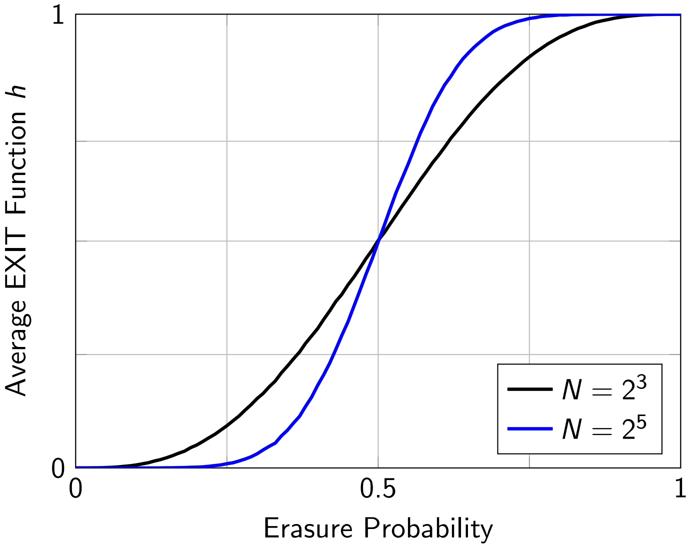

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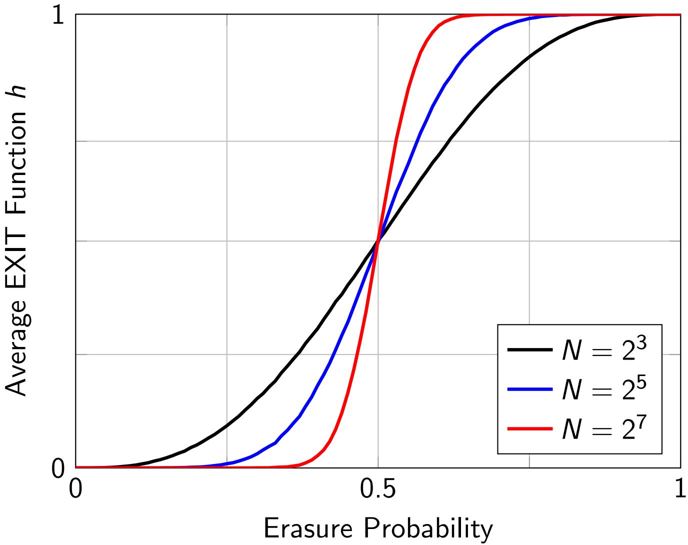

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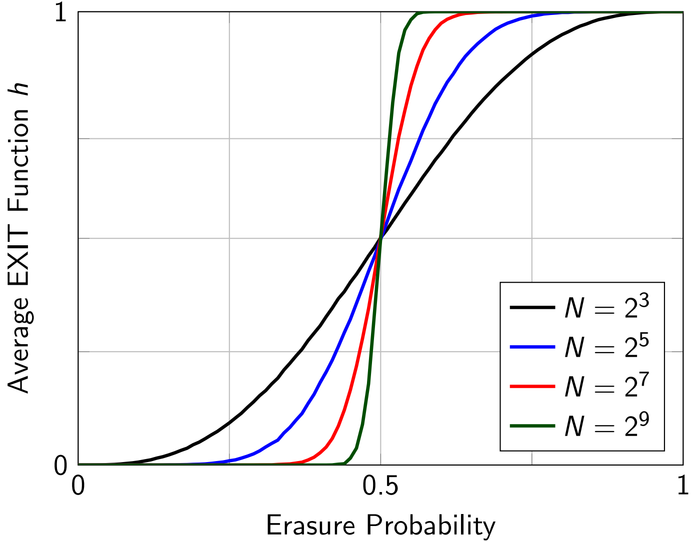

When do EXIT Functions Exhibit $0 - 1$ *Transition?*

EXIT Function as Measure of a Set Ω_i

Set of bad erasures that prevent recovery of X_i from $\underline{Y}_{\sim i}$

$$\Omega_i \triangleq \{\underline{z}_{\sim i} \in \{0,1\}^{N-1} \mid \exists\, \underline{x} \in \mathcal{C}, x_i = 1, \underline{x}_{\sim i} \leq \underline{z}_{\sim i}\}$$

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$$\mu_p(\Omega) \triangleq \sum_{\underline{a} \in \Omega} p^{|\underline{a}|}(1-p)^{N-1-|\underline{a}|}$$

EXIT Function and Monotone Boolean Functions

$$h_i(p) = \mu_p(\Omega_i)$$

$\Omega_i \longleftrightarrow$ Set of bad erasures

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Path Ahead: Symmetric Monotone Boolean Functions
Exhibit Sharp $0-1$ Transitions

Avg. EXIT Function h, not h_i, satisfies Area Theorem

Bit-i EXIT Function h_i, not h, is Monotone Boolean

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What about symmetry?

Group Symmetry

The Permutation Group $\mathcal{G}$ of code $\mathcal{C}$ is defined as

$$\mathcal{G} = \{\pi \in S_N \mid \pi(\underline{x}) \in \mathcal{C} \quad \forall\ \underline{x} \in \mathcal{C}\}$$

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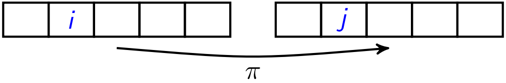
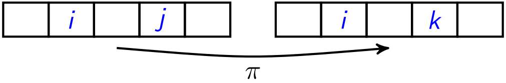
Transitive Permutation Groups

- $\mathcal{G}$ is transitive if for all i, j, $\exists\, \pi \in \mathcal{G}$ such that $\pi(i) = j$



- $\mathcal{G}$ is doubly transitive if for all distinct i, j, k, $\exists\, \pi \in \mathcal{G}$ such that $\pi(i) = i$ and $\pi(j) = k$



EXIT Functions Under Group Symmetry

Proposition

- If $\mathcal{G}$ is transitive

$$h_i(p) = h_j(p) = h(p) \quad \text{for all } 0 \leq p \leq 1$$

- If $\mathcal{G}$ is doubly transitive

  Ω_i is invariant under a transitive permutation group

EXIT Functions Under Double Transitivity

Under double transitivity: $h_i = h$ and Ω_i is transitive

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Symmetric Monotone Boolean Functions Exhibit
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Avg. EXIT Function h must exhibit a sharp $0-1$ transition!

(Symmetric) Monotone Boolean Functions invariant under Transitive Permutation Group

Bernoulli(p) Product Measure μ_p on $\{0,1\}^N$

$$f\colon \{0,1\}^N \to \{0,1\}, \qquad h(p) = \mathbb{E}_{\mu_p}[f].$$

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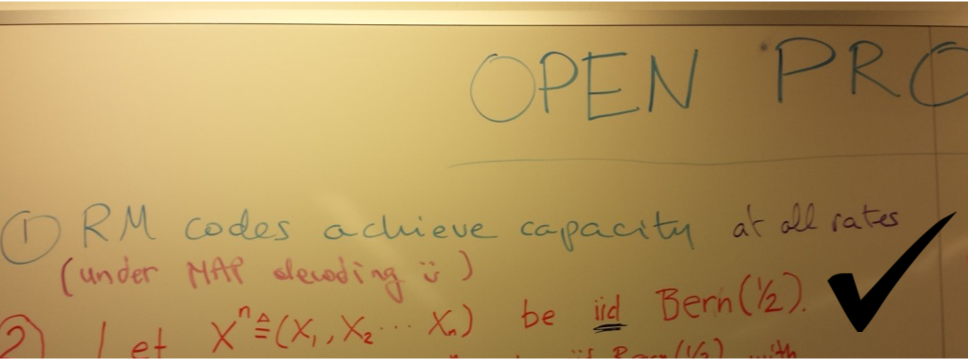
Popular Theorem in TCS

- Shown in early 1990s
- By Friedgut-Kalai, Talagrand, Bourgain-Kahn-Kalai-Linial
- Below, $p_t = h^{-1}(t)$

$$p_{1-\varepsilon} - p_\varepsilon \le 2C \frac{\log \frac{1}{\varepsilon}}{\log N}, \qquad p_{1-\varepsilon} - p_\varepsilon \to 0.$$



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Other Symmetric Monotone Boolean Functions

Monotone graph properties

(i) arguments to function indicate which edges present
(ii) invariance under relabeling of vertices gives symmetry

Hamming weight greater than r

Clearly symmetric and monotone

Capacity via Symmetry

Generality

- How general is this phenomenon?
- Proof heavily exploits MAP decoding on erasure channels
- Abbe et al. have shown for BSC when rate $\to 0$

Open Questions

- Extension to general BMS channels
- Practical decoders that achieve capacity for non-trivial rates
- Extension to rates converging to 0 or 1 (ala Friedgut)
- Capacity-achieving schemes for other systems?
  - Quantum codes, Rate-Distortion, Compressed Sensing